CHAPTER 3

On Fourier Transforms and Delta Functions

The Fourier transform of a function (for example, a function of time or space) provides a way to analyse the function in terms of its sinusoidal components of different wavelengths. The function itself is a sum of such components.

The Dirac delta function is a highly localized function which is zero almost everywhere. There is a sense in which different sinusoids are orthogonal. The orthogonality can be expressed in terms of Dirac delta functions.

In this chapter we review the properties of Fourier transforms, the orthogonality of sinusoids, and the properties of Dirac delta functions, in a way that draws many analogies with ordinary vectors and the orthogonality of vectors that are parallel to different coordinate axes.

3.1 Basic Analogies

If $\mathbf{A}$ is an ordinary three-dimensional spatial vector, then the component of $\mathbf{A}$ in each of the three coordinate directions specified by unit vectors $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, $\hat{\mathbf{x}}_3$ is $\mathbf{A} \cdot \hat{\mathbf{x}}_i$ for $i = 1, 2$, or 3. It follows that the three cartesian components (A_1, A_2, A_3) of $\mathbf{A}$ are given by

$$A_i = \mathbf{A} \cdot \hat{\mathbf{x}}_i, \qquad \text{for } i = 1, 2, \text{ or } 3. \tag{3.1}$$

We can write out the vector $\mathbf{A}$ as the sum of its components in each coordinate direction as follows:

$$\mathbf{A} = \sum_{i=1}^{3} A_i \hat{\mathbf{x}}_i. \tag{3.2}$$

Of course the three coordinate directions are orthogonal, a property that is summarized by the equation

$$\hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j = \delta_{ij}. \tag{3.3}$$

Fourier series are essentially a device to express the same basic ideas (3.1), (3.2) and (3.3), applied to a particular inner product space.

An *inner product space* is a vector space in which, for each two vectors f and g, we define a scalar that quantifies the concept of "a scalar equal to the result of multiplying f and g together." Thus, for ordinary spatial vectors $\mathbf{x}$ and $\mathbf{y}$ in three dimensions, the usual scalar product $\mathbf{x} \cdot \mathbf{y}$ is an inner product. In the case of functions $f(x)$ and $g(x)$ that may be represented by Fourier series over a range of x values such as $-\frac{1}{2}L \leq x \leq \frac{1}{2}L$, we can define an inner product $f.g$ by $\int_{-\frac{1}{2}L}^{\frac{1}{2}L} f.g^* \, dx$. Here, g^* is the complex conjugate of g. By analogy with ordinary vectors we can think of $\sqrt{f.f}$ as the real-valued positive scalar "length" of f, where now $\sqrt{f.f} = \sqrt{\int_{-\frac{1}{2}L}^{\frac{1}{2}L} f.f^* \, dx}$. Even if $f(x)$ is not a real function, $f.f^*$ is real and therefore the length $\sqrt{f.f}$ is real.

In the case of Fourier series, we consider a space consisting of functions that can be represented by their components in an infinite number $n = 1, 2, 3, \ldots$ of "coordinate directions," each one of which corresponds to a particular sinusoid.

Fourier transforms take the process a step further, to a *continuum* of n-values.

To establish these results, let us begin to look at the details first of Fourier series, and then of Fourier transforms.

3.2 Fourier Series

Consider a periodic function $f = f(x)$, defined on the interval $-\frac{1}{2}L \leq x \leq \frac{1}{2}L$ and having $f(x + L) = f(x)$ for all $-\infty < x < \infty$. Then the complex Fourier series expansion for f is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{2i\pi nx/L}. \tag{3.4}$$

First we define

$$I(l) = \int_{-\frac{1}{2}L}^{\frac{1}{2}L} e^{2i\pi lx/L} \, dx.$$

Then

$$I(l) = \frac{L}{2i\pi l}(e^{i\pi l} - e^{-i\pi l})$$

for $l \neq 0$. If l is an integer, $e^{i\pi l} = e^{-i\pi l}$ and $I(l) = 0$. If $l = 0$, then $I(0) = \int_{-\frac{1}{2}L}^{\frac{1}{2}L} 1 \, dx = L$. It follows that

$$\int_{-\frac{1}{2}L}^{\frac{1}{2}L} e^{2i\pi(n-m)x/L} \, dx = L \, \delta_{nm}, \tag{3.5}$$

and we can find the coefficients in (3.4) by multiplying (3.4) through by $e^{-2i\pi mx/L}$ and integrating over x from $-\frac{1}{2}L$ to $+\frac{1}{2}L$. Here we can work with the inner product space specified in Section 3.1. We can think of (3.5) as giving the component of the function

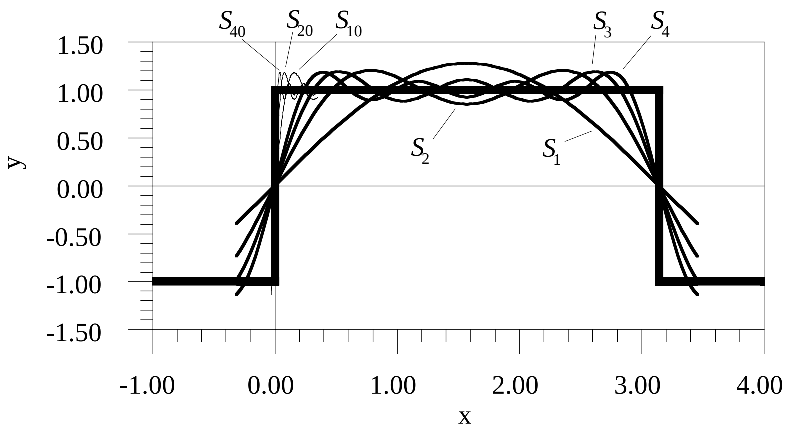

FIGURE **3.1**
The values of $f(x)$ are shown from $x = -1$ to $x = 4$ together with S_1, S_2, S_3, and S_4 as heavy lines from $x = -\pi/10$ to $x = 11\pi/10$, and S_{10}, S_{20}, and S_{40} as lighter lines from $x = -\pi/100$ to $x = 11\pi/100$. Further detail is given with an expanded scale in the next Figure.

$e^{2i\pi nx/L}$ in the direction of the function $e^{2i\pi mx/L}$. In application to (3.4), we find

$$\int_{-\frac{1}{2}L}^{\frac{1}{2}L} f(x)\, e^{-2i\pi mx/l}\, dx = Lc_m, \tag{3.6}$$

which determines the coefficients c_m in (3.4).

Comparing the last six equations, we see that (3.4), (3.5) and (3.6) correspond to (3.2), (3.3), and (3.1) respectively.

3.2.1 GIBBS' PHENOMENON

When a function having a discontinuity is represented by its Fourier series, there can be an "overshoot." The phenomenon, first investigated thoroughly by Gibbs, is best described with an example.

Thus, consider the periodic function

$$\begin{aligned} f(x) &= 1 \qquad \text{for } 0 < x < \pi \\ &= -1 \qquad \text{for } \pi < x < 2\pi \\ f(x + 2\pi) &= f(x) \qquad \text{for all } x. \end{aligned} \tag{3.7}$$

It has discontinuities at 0, $\pm\pi$, $\pm 2\pi$, $\pm 3\pi$, …, and the Fourier series for f is

$$f(x) = \frac{4}{\pi}\Big(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\Big). \tag{3.8}$$

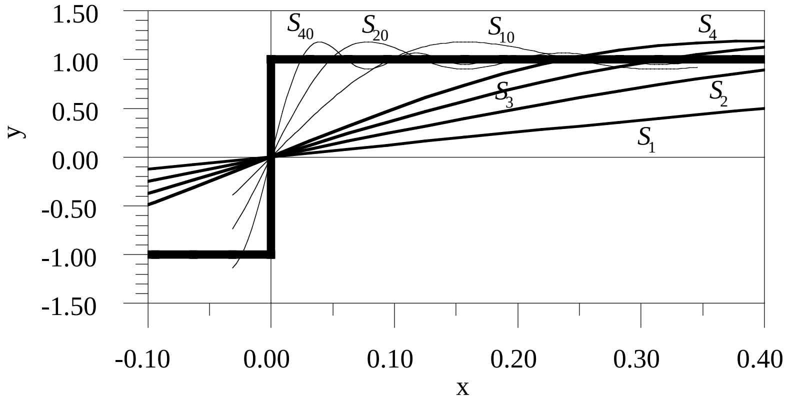

FIGURE **3.2**
Same as Figure 3.1, but with a scale expanded by a factor of 10 to show detail in the vicinity of a discontinuity.

To examine the convergence of this series, define

$$S_n(x) = \frac{4}{\pi} \sum_{j=1}^{n} \frac{\sin(2j-1)x}{2j-1}. \tag{3.9}$$

Then S_n is the sum of the first n terms of the Fourier series (3.8). Figure 3.1 gives a comparison between f and seven different approximations $S_n(x)$.

Clearly the $S_n(x)$ become better approximations to $f(x)$ as n increases. But even when n is quite large ($n = 10, 20, 40$) the series approximations overshoot the discontinuity. Figure 3.2 gives a close-up view.

Instead of jumping up from -1 to $+1$, the finite series approximations $S_n(x)$ for large n overshoot to values almost equal to ± 1.2. It turns out that the overshoot stays about the same, tending to about ± 1.18 in the limit as $n \to \infty$. The overshoot amounts to 18%! This means that Fourier series may not be very good for representing discontinuities. But they are often very good for respresenting smooth functions.

3.3 Fourier Transforms

We begin with

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{2i\pi nx/L} \tag{3.4 again}$$

where

$$c_n = \frac{1}{L} \int_{-\frac{1}{2}L}^{\frac{1}{2}L} f(x)\, e^{-2i\pi nx/L}\, dx \tag{3.6 again}$$

and let $L \to \infty$.

Define

$$k = \frac{2\pi n}{L} \quad \text{and} \quad Lc_n = g(k).$$

Then

$$\frac{L}{2\pi}\, dk = dn = 1$$

for unit increments in the summation (3.4), and this summation converts to an integral via

$$f(x) = \sum_{n=-\infty}^{\infty} (Lc_n) e^{2i\pi nx/L}\, \frac{dn}{L} = \int_{-\infty}^{\infty} g(k) e^{ikx}\, \frac{dk}{2\pi} \tag{3.10}$$

where

$$g(k) = \lim_{n\to\infty} Lc_n = \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx. \tag{3.11}$$

We refer to $g(k)$ as "the Fourier transform of $f(x)$." Equivalently, we say that $f(x)$ and $g(k)$ are "Fourier transform pairs."

Obviously, (3.10) corresponds to (3.4), and (3.11) corresponds to (3.6). But what about orthogonality?

Replacing k in (3.10) by K to get

$$f(x) = \int_{-\infty}^{\infty} g(K) e^{iKx}\, \frac{dK}{2\pi}$$

and then substituting this expression for $f(x)$ into (3.10), we obtain

$$g(k) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} g(K) e^{iKx}\, \frac{dK}{2\pi} \right] e^{-ikx}\, dx.$$

Re-arranging this,

$$g(k) = \int_{-\infty}^{\infty} g(K) \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(K-k)x}\, dx \right\} dK.$$

Since this last result is true for any $g(k)$, it follows that the expression in the big curly brackets is a Dirac delta function:

$$\delta(K-k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{i(K-k)x}\, dx. \tag{3.12}$$

This is the orthogonality result which underlies our Fourier transform. It says that e^{ikx} and e^{iKx} are orthogonal unless $k = K$ (in which case they are the same function). We discuss delta functions further in Section 3.4.

Comparing with Section 3.1: equations (3.10), (3.11), and (3.12) correspond to (3.2), (3.1) and (3.3) respectively.

3.3.1 THE UNCERTAINTY PRINCIPLE

This subsection describes an important attribute of Fourier transform pairs, namely that if one of the pair of functions has values that are large over only a limited range of its independent variable (x, say), then the Fourier-transformed function will have significant amplitude over a wide range of its independent variable (k, say). And if the transformed function is significant over only a narrow range of k values, the original function will be spread over a wide range of x values.

To appreciate these concepts, we shall work with the function

$$\begin{aligned} f(t) &= \mathrm{e}^{-t/T} \sin \Omega t \quad \text{if } \ 0 \le t \\ &= 0 \qquad \text{if } \ t < 0. \end{aligned} \tag{3.13}$$

which has the Fourier transform

$$\begin{aligned} f(\omega) &= \int_0^{\infty} \frac{e^{[i(\omega+\Omega)-1/T]t} - e^{[i(\omega-\Omega)-1/T]t}}{2i}\, dt \\ &= \frac{1}{2(\omega+\Omega)+2i/T} - \frac{1}{2(\omega-\Omega)+2i/T}. \end{aligned} \tag{3.14}$$

Note that in equation (3.14) we are working with the independent variables t and ω rather than with x and k. The sign convention is discussed in Box 3.1. Note too that we have chosen to use the same symbol, f, for the function under consideration, whether it is specified in the time domain as $f(t)$ in (3.13), or in the frequency domain as $f(\omega)$ in (3.14). This use of the same symbol may at first appear confusing because previously we worked with the concept of two functions, for example $f(x)$ and $g(k)$ when we were considering a function of x and its transform in the k domain. The reason for now using the same symbol is to acknowledge a deep truth, namely that we are really working with information, and it doesn't matter whether we express this information in the time domain or in the frequency domain. Thus, in terms of the information contained in a function such as $f(t)$, when we transform it to the frequency domain we have simply chosen to use a different way to look at the same information that was contained in the original function. Because it is really

BOX **3.1**

Sign Conventions, and Multi-dimensional Transforms

In equation (3.14), note that the sign convention taken for the transform of $f(t)$ is $f(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt$, whereas previously we used $f(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$ for the spatial transform. The reality is that numerous different conventions are in use. Therefore, when working out the details of a particular application of Fourier analysis, it is important to be sure you know what the convention is. It is quite common to do what we are doing here, namely, use a different convention for spatial transforms, than for time transforms. Throughout these notes, we use

$$f(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt, \quad \text{and} \quad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)e^{-i\omega t}\,d\omega \tag{1}$$

for the time transform, and

$$f(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx, \quad \text{and} \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(k)e^{+ikx}\,dk \tag{2}$$

for a spatial transform.

In the second of equations (1) we have written $f(t)$ as a summation of its frequency components; and in the second of equations (2), $f(x)$ is a summation of its wavenumber components.

Spatial transforms may have additional dimensions, such as the 3D transform from $\mathbf{x} = (x_1, x_2, x_3)$ to $\mathbf{k} = (k_1, k_2, k_3)$ expressed by

$$f(\mathbf{k}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{x})\, e^{-i\mathbf{k}\cdot\mathbf{x}}\,dx_1\,dx_2\,dx_3 \tag{3a}$$

where $\mathbf{k}\cdot\mathbf{x} = k_1x_1 + k_2x_2 + k_3x_3$. The reverse transform, in which $f(\mathbf{x})$ is represented as a summation of its wavenumber components, is

$$f(\mathbf{x}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}}\,dk_1\,dk_2\,dk_3. \tag{3b}$$

An example of a combined space-time transform, from (x_1, x_2, t) to (k_1, k_2, ω), is used in Section 2.1.4. It has the form

$$f(k_1, k_2, x_3, \omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{x}, t)\, e^{-i(k_1x_1+k_2x_2-\omega t)}\,dx_1\,dx_2\,dt \tag{4a}$$

and

$$f(\mathbf{x}, t) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(k_1, k_2, x_3, \omega)\, e^{i(k_1x_1+k_2x_2-\omega t)}\,dk_1\,dk_2\,d\omega. \tag{4b}$$

This pair of equations is very useful, when it is possible to obtain a specific form for the transformed solution $f(k_1, k_2, x_3, \omega)$. In Section 2.1.4 we found that an integrand equivalent to (4b) has the form of a plane wave, and hence that such basic solutions can be summed to provide more general solutions to the wave equation.

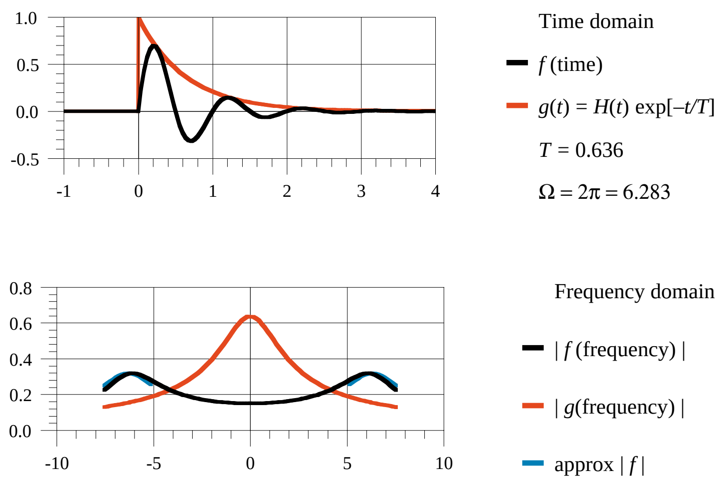

FIGURE **3.3**
$f(t)$ and its amplitude spectrum $|f(\omega)|$. Values of T and Ω are indicated, and $T\Omega = 4$ in this case. The approximate amplitude spectrum is given for ω in the range $\pm 20\%$ around $\omega = -\Omega$ and $\omega = \Omega$, based on the formulas in (3.15). Also shown is the function $H(t)\exp(-t/T)$ and its spectrum. Note the highest values of $|f(\omega)|$ occur at $\omega = \pm\Omega$, though these peaks in the spectrum are not very sharp in this case.

the same information, albeit displayed in a different way, it makes sense to use the same symbol, f. To remind ourselves of how the information is displayed, we refer to it either as $f(t)$ or $f(\omega)$.

To illustrate the "uncertainty principle," we return to a discussion of equations(3.13) and (3.14). An example of $f(t)$ and its amplitude spectrum $f(\omega)$ for a particular choice of T and Ω is shown in Figure 3.3.

We begin by noting that $f(\omega)$ has poles at $\omega = \pm\Omega + i/T$, which are near the real ω axis if T is large. It follows that an approximation to $f(\omega)$ is given by

$$\begin{aligned} \text{approx } f(\omega) &= \frac{1}{2(\omega+\Omega)+2i/T} \qquad \text{for } \omega \text{ near } -\Omega \\ &= \frac{-1}{2(\omega-\Omega)+2i/T} \qquad \text{for } \omega \text{ near } +\Omega. \end{aligned} \tag{3.15}$$

The power spectrum of f, defined as $|f(\omega)|^2$, (i.e. the square of the amplitude spectrum) is therefore concentrated near the two frequency values $\omega = \pm\Omega$ if T is large, and is then approximated by

$$\frac{1}{4[(\omega+\Omega)^2+1/T^2]} \text{ for } \omega \text{ near } -\Omega$$

$$\frac{1}{4[(\omega-\Omega)^2+1/T^2]} \text{ for } \omega \text{ near } \Omega,$$

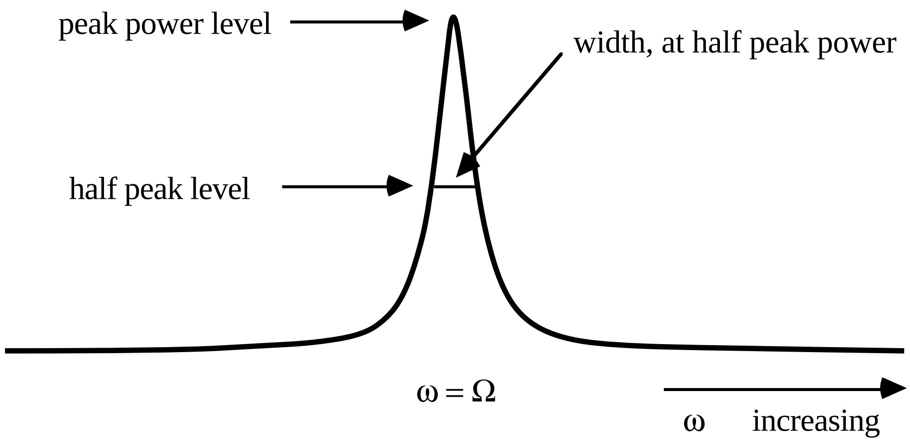

FIGURE **3.4**
An illustration defining the width of a peak at the half-power level. The peak here is centered at $\omega = \Omega$.

which each have peaks of height $T^2/4$.

It follows that the value of $|f(\omega)|^2$ at $\omega = \Omega \pm 1/T$ has half the power of the maximum located at $\omega = \Omega$. The width of the region of significant values of $|f(\omega)|^2$, concentrated near $\omega = \Omega$, is therefore $2/T$ if the width is measured at ω values where the power level has dropped to half its maximum. The definition of the "width at half power" is shown in Figure 3.4.

The width of the original function $f(t)$ is effectively T. So we see that the product of the width of the range of t values where $f(t)$ is significant, times the width of the range of ω values where $f(\omega)$ is significant, is constant. As one width is increased, the other must shrink. This is the result we refer to as the uncertainty principle for Fourier transforms: we cannot obtain information confined over a short range of t values, that is also confined over a short range of ω values.

Going on to see how the principle is expressed in practice for the functions $f(t)$ and $f(\omega)$ of equations (3.13) and (3.14) with different values of T, Figure 3.5 shows a situation where information is spread out in time and concentrated in frequency. Figure 3.6 shows the opposite situation — an example where information is restricted in time and spread out in frequency.

Figures 3.3, 3.5 and 3.6 also show $g(t) = H(t)e^{-t/T}$ and its amplitude spectrum, which is

$$|g(\omega)| = \frac{1}{\sqrt{\omega^2 + 1/T^2}}, \text{ based on } g(\omega) = \frac{i}{\omega + i/T}. \tag{3.16}$$

If $1/T \ll \omega$, (3.16) gives $g(\omega) = 1/(-i\omega)$, which in turn implies in the time domain that $g(t) = H(t)$, the unit step function. This is the situation shown in Figure 3.5. If $|\omega| \ll 1/T$, (3.16) gives $g(\omega) = T$, which in turn implies in the time domain that $g(t) = T\delta(t)$. This is the situation shown in Figure 3.6. These two results are examples of the general idea that

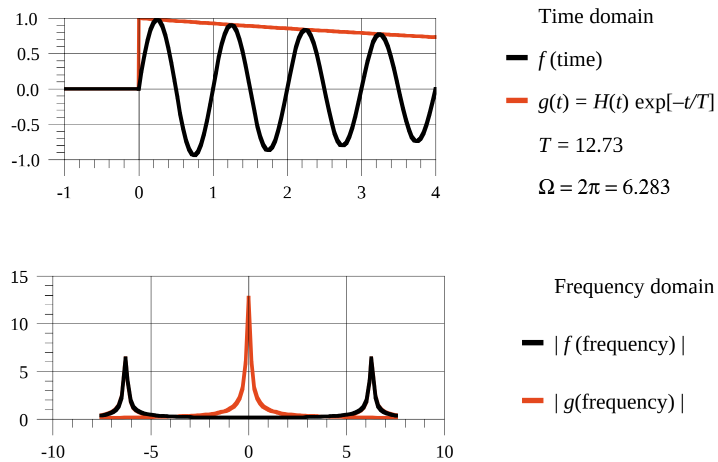

FIGURE **3.5**
$f(t)$ and its amplitude spectrum $|f(\omega)|$, for $T\Omega = 80$. The amplitude spectrum is strongly peaked at $\omega = -\Omega$ and $\omega = \Omega$. The approximation given in (3.15) is very accurate in this case and gives a result (not shown here) that is indistinguishable from the exact spectrum. Information is widely spread out in the time domain. (The range of t is shown, for only a fraction of the range for which $f(t)$ has significant values). Note also that the function $g(t)$ is approximately the unit step function $H(t)$ in this case. Its amplitude spectrum is approximately $1/\omega$. The dependence of $|g(\omega)|$ on a power of ω, namely on ω^{-1}, is hard to see in this Figure but can be easily made apparent if we plot $\log|g(\omega)|$ versus $\log\omega$, because then the values of $\log|g(\omega)|$ would fall on a straight line of slope -1. Amplitude spectra are often shown with log–log scales in order to reveal underlying power-law dependences.

if we look at the properties of a function of time over very long time scales, then in the frequency domain these properties are apparent from the spectrum at very low frequencies — and *vice versa*: properties of $g(t)$ over short time scales (for example, it may have a step discontinuity) are also apparent from study of $g(\omega)$ at high frequency (in the present case, behavior like $1/(-i\omega)$).

3.3.2 A FUNCTION WHOSE SHAPE IS SIMILAR TO THE SHAPE OF ITS FOURIER TRANSFORM

In statistics we often use so-called Gaussian curves, of the form

$$f(x) = \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}. \tag{3.17}$$

The width of the Gaussian is controlled by σ. For f as a probability density, $f(x)\,dx$ would be the probability of f lying between x and $x + dx$. Since $\int_{-\infty}^{\infty} e^{-\lambda x^2}\,dx = \sqrt{\pi/\lambda}$ (see Box 3.2), there is unit area under the function given in (3.17), indicating that the probability of f lying between $-\infty$ and $+\infty$ is 1, as we would expect for any probability density.

The Fourier transform of $f(x)$, namely $g(k)$, is given by

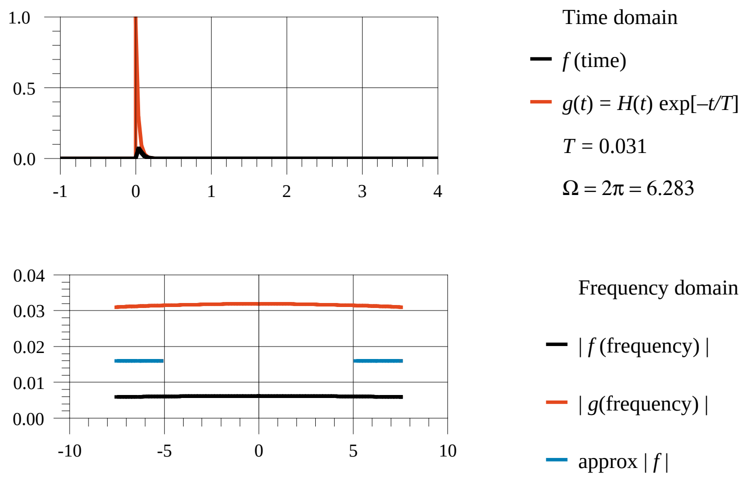

FIGURE **3.6**
$f(t)$ and its amplitude spectrum $|f(\omega)|$, for $T\Omega = 0.2$. The amplitude spectrum is spread out over a wide range of ω values (it is shown here, and in two previous Figures, only for the range $-1.2 \times \Omega < \omega < 1.2 \times \Omega$). The approximation for $|f(\omega)|$ given in equation (3.15) is very poor, since the poles at $\omega = \pm\Omega + i/T$ are not near the real ω axis. In the time domain, information is limited to only a short range of t values. Note that the function $g(t)$ is approximately $T \times \delta(t)$ in this case. Its amplitude spectrum is approximately a constant value, T.

$$
\begin{aligned}
g(k) &= \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx = \int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}\, e^{-ikx}\, dx \\
&= \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{\sigma^2k^2}{2}} \int_{-\infty}^{\infty} e^{-(\frac{x}{\sqrt{2}\sigma}+\frac{i\sigma k}{\sqrt{2}})^2}\, dx \\
&= \frac{1}{\sqrt{\pi}}\, e^{-\frac{\sigma^2k^2}{2}} \int_{-\infty+\frac{i\sigma k}{\sqrt{2}}}^{\infty+\frac{i\sigma k}{\sqrt{2}}} e^{-\mu^2}\, d\mu \\
&= \frac{1}{\sqrt{\pi}}\, e^{-\frac{\sigma^2k^2}{2}} \int_{-\infty}^{\infty} e^{-\mu^2}\, d\mu \\
&= e^{-\frac{\sigma^2k^2}{2}}.
\end{aligned} \tag{3.18}
$$

(Additional details of the above evaluation are given in Section 6.1.3, in the context of evaluating a similar integral, given in (6.28).)

We see here that the Fourier transform of a Gaussian is itself a Gaussian, but now the width of the transformed Gaussian is controlled by $1/\sigma$. A number of examples of f and g are shown in Figure 3.7 with different values of σ.

We note that $g(0) = 1$. We should expect this from a remark made above, about the area under the Gaussian curve (3.17) being unity, because the area under $f(x)$ is given by $\int_{-\infty}^{\infty} f(x)\, dx$ and in general this is just $g(k)$ evaluated for $k = 0$ (see the first of equations (3.18) with $k = 0$). So, the area under a function is equal to the long wavelength

BOX **3.2**

The area under a Gaussian curve

To prove

$$\int_{-\infty}^{\infty} e^{-\lambda x^2} dx = \sqrt{\frac{\pi}{\lambda}} \qquad (1)$$

we first consider the case $\lambda = 1$ and define $I = \int_{-\infty}^{\infty} e^{-x^2} dx$. Then

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \iint_{\text{whole } x-y \text{ plane}} e^{-(x^2+y^2)} dx\, dy.$$

But, integrating over the whole xy plane can be done using cylindrical coordinates (r, ϕ) where $r\cos\phi = x$, $r\sin\phi = y$, and $r^2 = x^2 + y^2$. Since the integrand for I^2 depends only upon r we can integrate over the whole $x - y$ plane by summing contributions from concentric area elements $2\pi r dr$, and $I^2 = 2\pi \int_{-\infty}^{\infty} e^{-r} r\ dr$. With $s = r^2$, $I^2 = \pi \int_0^{\infty} e^{-s} ds = \pi \left.\frac{e^{-s}}{-1}\right|_{s=0}^{s=\infty} = \pi$, and so $I = \sqrt{\pi}$. To prove (1), we simply replace $\sqrt{\lambda}x$ by y. Then

$$\int_{-\infty}^{\infty} e^{-\lambda x^2} dx = \int_{-\infty}^{\infty} e^{-y^2} dx/\sqrt{\lambda} = I\sqrt{\lambda} = \sqrt{\pi/\lambda}.$$

or long period limit of the spectrum (using terminology appropriate for functions of space or time). In the present case, the area under the original curve (3.17) is unity, therefore we must have $g(0) = 1$.

We see from Figure 3.7 that the width of the function in the x-domain goes up as σ increases, and the width of the function in the k-domain correspondingly goes down. This is another example of the general point made in the previous subsection, about a trade-off in the way that information is concentrated in one domain or the other.

3.4 More on Delta Functions

Let us go back to the "substitution" property of the Kronecker delta function, defined in Chapter 1. From the first of (1.8), this property is

$$A_i \delta_{ij} = A_j. \qquad (3.19)$$

One way to think of the summation contained in the left-hand side here, is that it represents a *weighted average* of all the different A_i values $(i = 1, 2, 3)$. Because $\delta_{ij} = 0$ for $i \neq j$, the only contributing term is the one for which $i = j$. The summation over i, then gives the right-hand side value A_j.

When we investigated Fourier series in Section 3.2 we obtained a generalization of the above, in that

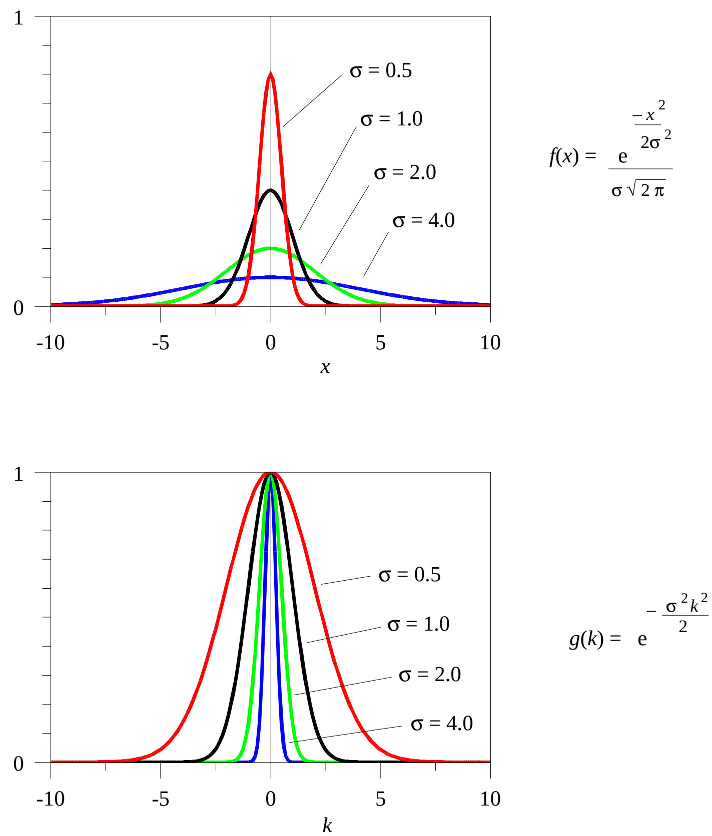

FIGURE **3.7**
$f(x)$ (top) and its spectrum $g(k)$ (bottom), from equations (3.17) and (3.18), for four different values of σ. If $\sigma = 1$, the original function and its transform have the same half-width. But for any other value of σ, one function has a half-width wider than the case $\sigma = 1$ and the other function has a half-width that is narrower.

$$\frac{1}{L}\int_{-\frac{1}{2}L}^{\frac{1}{2}L} e^{2i\pi(n-m)x\pi\ L}\,dx = \delta_{mn} \tag{3.20}$$

and this again is a Kronecker delta function (equal to 0 for $m \neq n$; equal to 1 for $m = n$). But now $m = 1, 2, \ldots, \infty$ and $n = 1, 2, \ldots, \infty$. So now the inner product space has a countably infinite number of dimensions.

Again, we have the substitution property

$$\sum_{m=1}^{\infty} c_m \delta_{mn} = c_n. \tag{3.21}$$

Paul Dirac took a major step in generalizing (3.19) and (3.21) to develop what today we call the Dirac delta function $\delta(x - X)$, with the property

$$\int_{-\infty}^{\infty} f(x)\,\delta(x - X)\,dx = f(X). \tag{3.22}$$

This delta function is again a function of two variables, but now they are two continuous variables (x and X), instead of the integer variables i and j of (3.19), or integer variables m and n of (3.21). We should not let such superficial differences in notation obscure the fact that there are strong similarities between (3.19), (3.21), and (3.22).

Another way to convey these results, is to note that the defining properties of a one-dimensional Dirac delta function $\delta(x, X)$ are that

$$\begin{aligned} \delta(x, X) = 0 \qquad &\text{if } x \neq X, \text{ and} \\ \int_{-\infty}^{\infty} \delta(x, X)\,dx = 1. & \end{aligned} \tag{3.23}$$

It is only a notation convention, that we write δ_{ij} rather that $\delta(i - j)$; and usually we write $\delta(x - X)$ rather that δ_{xX} or $\delta(x, X)$. All these delta functions are mostly equal to zero, and have non-zero value only where the difference in independent variables (i and j, m and n, x and X) is equal to zero.

All of (3.19), (3.21), and (3.22) are examples of weighted averaging, in which the weights are so strong for one particular value ($i = j, m = n, x = X$), and so weak for all other values ($i \neq j, m \neq n, x \neq X$), that only one value of the original function has any importance (A_i for $i = j$; c_m for $m = n$; $f(x)$ for $x = X$).

What is particularly strange about the Dirac delta function $\delta(x - X)$, is that it is zero *everywhere* as a function of x (provided $x \neq X$); but at $x = X$ itself, it has such a strong value (super infinite), that it gives a finite result when we do the integration in (3.22). These properties of the Dirac delta function were for many years a challenge that was associated with new developments in the theory of integration of generalized functions. Delta functions of a continuous variable can be thought of as the limit, as $\varepsilon \to 0$, of a sequence of functions like

$$\begin{aligned} B(x, X, \varepsilon) &= 0 \qquad &&\text{for } x < X - \tfrac{1}{2}\varepsilon \\ &= \frac{1}{\varepsilon} &&\text{for } X - \tfrac{1}{2}\varepsilon \le x \le X + \tfrac{1}{2}\varepsilon \\ &= 0 &&\text{for } X < x, \end{aligned} \tag{3.24}$$

or

$$G(x, X, \varepsilon) = \frac{1}{\sqrt{2\pi}\,\varepsilon} e^{\frac{-(x-X)^2}{2\varepsilon^2}}. \tag{3.25}$$

At any fixed value of ε, $B(x, X, \varepsilon)$ and $G(x, X, \varepsilon)$ have unit area in the sense that

$$\int_{-\infty}^{\infty} B(x, X, \varepsilon)\,dx = \int_{-\infty}^{\infty} G(x, X, \varepsilon)\,dx = 1.$$

Also,

$$\lim_{\varepsilon\to 0} B(x, X, \varepsilon) = \lim_{\varepsilon\to 0} G(x, X, \varepsilon) = 0 \text{ for } x \neq X.$$

These functions, B and G, are two of many such functions that can be used to establish the basic properties of the Dirac delta function.

In Section 6.1.3 and Box 6.1, we discuss additional features of Dirac delta functions in time and space.

Suggestions for Further Reading

Snieder, Roel. *a Guided Tour of Mathematical Methods for the Physical Sciences*, pp 186–194 for delta functions, pp 200–211 for Fourier analysis. Cambridge, UK: Cambridge University Press, 2001.

Problems

3.1 In the case that $f(t)$ is real, the Fourier transform $f(\omega)$ defined for example by equation (1) of Box 3.1 is subject to constraints that allow us to avoid the use of negative frequencies, and to work with the real part of $f(\omega)$ alone, rather than with the complex-valued transform, $f(\omega)$. The reader is asked to obtain the main results as follows, when $f(t)$ is real:

a) If $f(\omega)$ is expressed in terms of its real and imaginary parts by $f(\omega) = \Re[f(\omega)] + i\Im[f(\omega)]$, show that $\Re[f]$ is even in ω, and $\Im[f]$ is odd. That is, show

$$\Re[f(-\omega)] = +\Re[f(\omega)] \quad \text{and} \quad \Im[f(-\omega)] = -\Im[f(\omega)].$$

b) Show that $\Im[f(t)] = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} (\Im[f(\omega)]\cos\omega t - \Re[f(\omega)]\sin\omega t)\ d\omega$.

c) Show that result a), applied to result b), does give $\Im[f(t)] = 0$ and that therefore a) and b) are consistent with $f(t)$ being purely real.

d) Show that $f(t) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} (\Re[f(\omega)]\cos\omega t + \Im[f(\omega)]\sin\omega t)\ d\omega$.

e) Hence show that

$$f(t) = \frac{1}{\pi}\int_0^{\infty} \Re[f(\omega)]\cos\omega t\ d\omega.$$

[This is the key result that allows us to work with only with $0 \leq \omega$ when evaluating $f(t)$ as a summation over its frequency components.]