



Levi-flat CR structures on compact Lie groups

Howard Jacobowitz¹ · Max Reinhold Jahnke²

Received: 31 January 2023 / Accepted: 6 June 2023 / Published online: 21 June 2023
© The Author(s) 2023

Abstract

Pittie (Proc Indian Acad Sci Math Sci 98:117-152, 1988) proved that the Dolbeault cohomology of all left-invariant complex structures on compact Lie groups can be computed by looking at the Dolbeault cohomology induced on a conveniently chosen maximal torus. We generalized Pittie's result to left-invariant Levi-flat CR structures of maximal rank on compact Lie groups. The main tools we used was a version of the Leray–Hirsch theorem for CR principal bundles and the algebraic classification of left-invariant CR structures of maximal rank on compact Lie groups (Charbonnel and Khalgui in J Lie Theory 14:165-198, 2004).

Keywords CR structures · Tangential Dolbeault cohomology · Compact Lie groups

Mathematics Subject Classification 58J10 · 22C05 · 22E30

1 Introduction

In this work, we prove a version of the Leray–Hirsch theorem for CR principal bundles, which, combined with a result by Charbonnel and Kalgui [6], allows us to study the $\bar{\partial}_b$ cohomology of left-invariant Levi-flat CR structures of maximal rank on compact Lie groups.

Let G be a connected and compact Lie group with Lie algebra \mathfrak{g} and let $\mathbb{C}\mathfrak{g}$ be the complexification of \mathfrak{g} . We assume that G is odd-dimensional and is endowed with a left-invariant Levi-flat CR structure \mathcal{V} of CR codimension 1. Since \mathcal{V} is left-invariant, there is a corresponding Lie algebra $\mathfrak{h} \subset \mathbb{C}\mathfrak{g}$ defined by the restriction of \mathcal{V} to the identity of G . Notice that this is a one-to-one correspondence, that is, given a Lie subalgebra $\mathfrak{h} \subset \mathbb{C}\mathfrak{g}$, we define by left-translation an involutive vector bundle $\mathcal{V} \subset \mathbb{C}TG$ and if we assume that $\mathfrak{h} \cap \bar{\mathfrak{h}} = \{0\}$ the vector bundle \mathcal{V} is an abstract CR structure. Notice that the rank of the vector bundle \mathcal{V} is just the dimension of the Lie algebra \mathfrak{h} and that the commutator bracket and the Lie algebra

✉ Max Reinhold Jahnke
jahnkem@mathematik.uni-marburg.de

Howard Jacobowitz
jacobowi@camden.rutgers.edu

¹ Department of Mathematical Sciences, Rutgers University, Camden, NJ, USA

² Fachbereich Mathematik und Informatik, Philipps-Universität Marburg, Marburg, Hessen, Germany

bracket agree. Further, because we use induced CR structure on a maximal torus, we restrict the rank of G to be bigger than or equal to 3.

We consider the $\bar{\partial}_b$ complex on G

$$\bar{\partial}_b : C^\infty(G; \Lambda^{0,q}) \rightarrow C^\infty(G; \Lambda^{0,q+1})$$

with $0 \leq q \leq \text{rank } \mathcal{V}$ with cohomology spaces denoted by $H^{0,q}(G; \mathcal{V})$.

Given $\xi \in \mathfrak{h}^\perp = \{u \in \mathbb{C}\mathfrak{g}^* : u(L) = 0 \forall L \in \mathfrak{h}\}$, $\xi \neq 0$, the Levi form at $\xi \in \mathfrak{h}^\perp$ is the Hermitian form on \mathfrak{h} defined by $\mathcal{L}_\xi(L, M) = (1/2i)\xi([L, \bar{M}])$ in which L and M are any smooth local sections of \mathfrak{h} . Notice that the Levi-form is left-invariant. We shall say that \mathcal{L} is Levi-flat if given any point $\xi \in \mathfrak{h}^\perp$, with $\xi \neq 0$, the Hermitian form \mathcal{L}_ξ is zero.

By the classification theorem of Charbonnel and Kalgui [6], there is a maximal torus $T \subset G$ such that $\mathcal{W}_t \doteq \mathcal{V}_t \cap \mathbb{C}T_t T$ defines a bi-invariant CR structure $\mathcal{W} = \bigcup_{t \in T} \mathcal{W}_t$ on T . We call \mathcal{W} the toric component of \mathcal{V} (relative to the maximal torus T). For each $g \in G$, we denote by gT the set $\{gt : t \in T\}$ and by $g\mathcal{W}$ the pushforward by left-translation by g , that is, $g\mathcal{W} \doteq (L_g)_*(\mathcal{W})$ with $L_g(x) = gx$. We usually denote the toric component \mathcal{W} by its corresponding Lie algebra $\mathfrak{m} \subset \mathbb{C}\mathfrak{t}$ with \mathfrak{t} the Lie algebra of T .

We say that \mathcal{W} (or \mathfrak{m}) satisfies the divisor condition (DC) if there exist a basis $\{L_1, \dots, L_n\}$ for \mathcal{W} (or \mathfrak{m}) and constants $C, \rho > 0$ such that

$$\max_j |\widehat{L_j}(\xi)| \geq C(1 + |\xi|)^{-\rho}, \quad \forall \xi \in \mathbb{Z}^N,$$

with $\widehat{L_j}$ being the symbol of the vector field L_j and N the dimension of T .

The following theorem is the main result of this work.

Theorem 1 *Let G be a connected, odd-dimensional, and compact Lie group endowed with a left-invariant Levi-flat CR structure \mathcal{V} of maximal rank. Suppose that \mathcal{W} , the toric part of \mathcal{V} , satisfies the (DC) condition, then there exists an isomorphism $H^{0,q}(G; \mathcal{V}) \cong H^{0,q}(T; \mathcal{W})$.*

Notice that the toric part of left-invariant complex structures always satisfies the (DC) condition and so our theorem is a generalization, and a new proof, of a result by Pittie [14].

Here is a simple example to show that the hypothesis of Theorem 1 is not vacuous.

Example 1 The Lie algebra of $SU(2)$ is generated by the matrices

$$X = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

It is easy to verify that these matrices satisfy the following commutation relations:

$$[T, X] = 2Y, \quad [T, Y] = -2X, \quad [X, Y] = 2T. \tag{1}$$

Using coordinates (x, y) on \mathbb{T}^2 , we can easily verify that the involutive structure defined by $\mathfrak{h} \doteq \text{span}_{\mathbb{C}}\{\lambda\partial/\partial x + \partial/\partial y - iT, L\}$ is CR and Levi-flat. This follows a general technique found in [7]. Additionally, it can be shown that this structure satisfies the (DC) condition if λ is chosen as a non-Liouville number [10].

The proof of Theorem 1 is an adaptation to the CR case of a proof of the complex case we found in [1], and follows from the next two theorems: Theorem 2 is a version of Leray–Hirsch theorem for CR bundles assuming a cohomological condition and that the structure is Levi-flat, and Theorem 3 shows that (DC) condition implies such a cohomological condition.

Theorem 2 *Let G be a connected and compact Lie group endowed with a left-invariant Levi-flat CR structure \mathcal{V} . Let T and \mathcal{W} be as Theorem 1. A sufficient condition for the existence of an isomorphism for all q*

$$H^{0,q}(G; \mathcal{V}) \cong H^{0,q}(T; \mathcal{W})$$

is that for each q and for every cohomology class u in $H^{0,q}(T; \mathcal{W})$ there exists a cohomology extension u' in $H^{0,q}(G; \mathcal{V})$ such that for all $g \in G$, u' , when restricted to gT , defines a cohomology class $H^{0,q}(gT; g\mathcal{W})$.

The proof of this theorem can be obtained using the general Leray–Hirsch theorem [16, Theorem 9 of Section 7, Chapter 5]. The hypothesis that \mathfrak{h} is Levi-flat is used to show that the principal bundle G with structure group T and base space G/T admits a CR trivialization. Instead of just citing that reference, we prove Theorem 2 using only the theory of partial differential equations and complex analysis.

The next theorem shows that the (DC) condition on the toric component of the CR structure guarantees that a cohomology extension exists.

Theorem 3 *Let G be a connected and compact Lie group endowed with left-invariant CR structure \mathcal{V} of maximal rank, and let \mathcal{W} be its toric component. If \mathcal{W} satisfies the (DC) condition, then there exists a cohomology extension $H^{0,q}(T; \mathcal{W}) \rightarrow H^{0,q}(G; \mathcal{V})$ satisfying the conditions from Theorem 2.*

It follows from Theorem 3 that computation of the cohomology of Example 1 can be reduced to the maximal torus. We compute the dimension of the cohomology spaces in Example 6.

The paper is organized in the following way. In Sect. 2, we briefly introduce the notation regarding locally integrable structures, the associated differential complexes, and their cohomology spaces.

In Sect. 3, we define the main object of study of this work, namely the left-invariant CR structures of maximal rank. We also state the theorem of Charbonell and Kalgui [6] that gives a classification of such CR structures in the language of Lie algebras. We finish this section with examples of such structures.

In Sect. 4, we briefly recall some concepts related to principal and CR bundles and prove a version of the Leray–Hirsch theorem for CR bundles.

In Sect. 5, we discuss some examples to emphasize why some hypotheses are necessary and we also show that there are no Levi-flat CR structures of maximal rank on semisimple Lie groups.

In Sect. 6, we show that (DC) suffices to guarantee that cohomology class extensions exist.

In Sect. 7, we describe some open problems that would nicely complement our results.

2 Involutive structures

Let Ω be a smooth and orientable manifold of dimension N . An involutive structure on Ω is a smooth subbundle \mathcal{V} of the complexified tangent bundle $\mathbb{C}T\Omega$ of Ω such that the Lie bracket of any two smooth local sections of \mathcal{V} is again a smooth section of \mathcal{V} . We denote the rank of \mathcal{V} by n , and we denote by Λ^k the bundle $\Lambda^k \mathbb{C}T^*\Omega$. If W is a smooth vector bundle, we denote by $\mathcal{C}^\infty(\Omega; W)$ the space of sections of W with smooth coefficients.

For $0 \leq p \leq m \doteq N - n, 0 \leq q \leq n$ and each open set $U \subset \Omega$, there is a natural differential operator d' associated with \mathcal{V} defining a complex

$$d' : \mathcal{C}^\infty(U; \Lambda^{0,q}) \rightarrow \mathcal{C}^\infty(U; \Lambda^{0,q+1}) \tag{2}$$

with cohomology spaces denoted by $H^{0,q}(U; \mathcal{V})$. We refer to [2] and [17] for a detailed construction of this complex.

When $\mathcal{V} \oplus \bar{\mathcal{V}} = \mathbb{C}T\Omega$, the structure \mathcal{V} is called a *complex structure* and the differential operator d' is the usual $\bar{\partial}$ operator, and when $\mathcal{V} \cap \bar{\mathcal{V}} = \{0\}$, the structure \mathcal{V} is called a (*abstract*) *CR structure* and the differential operator d' denoted by $\bar{\partial}_b$.

3 Left-invariant involutive structures on compact Lie groups

Let G be a compact Lie group of dimension N . We denote by \mathfrak{g} the Lie algebra of G . If $\mathfrak{h} \subset \mathfrak{g}$ is any subalgebra, we denote by $\mathbb{C}\mathfrak{h}$ its complexification. Let $L_g : G \rightarrow G$ be the left-multiplication by $g \in G$, that is, $L_g(x) = g \cdot x$. Notice that the push-forward $(L_g)_*$ induces a vector bundle isomorphism between $G \times \mathbb{C}\mathfrak{g}$ and $\mathbb{C}TG$. Therefore, we can identify subbundles of $\mathbb{C}TG$ with subbundles of $G \times \mathbb{C}\mathfrak{g}$.

A subbundle $\mathcal{V} \subset \mathbb{C}TG$ is said to be left-invariant if $(L_g)_*X \in \mathcal{V}_{g \cdot x}$ for all $X \in \mathcal{V}_x$. We identify left-invariant involutive subbundles of $\mathbb{C}TG$ with Lie subalgebras of $\mathbb{C}\mathfrak{g}$. For a subalgebra $\mathfrak{h} \subset \mathbb{C}\mathfrak{g}$, we denote by $\mathcal{V}_\mathfrak{h}$ the associated subbundle. Notice that the rank of the bundle $\mathcal{V}_\mathfrak{h}$ is the complex dimension of \mathfrak{h} .

3.1 Left-invariant CR structures of maximal rank on compact Lie groups

In this section, we make a brief exposition of some results from [6].

Let $\mathfrak{t} \subset \mathfrak{g}$ be a maximal abelian subalgebra. We denote by Δ the set of roots of $\mathbb{C}\mathfrak{t}$ in $\mathbb{C}\mathfrak{g}$ and by Δ_+ a maximal subset of positive roots of Δ . For $\alpha \in \Delta$, we denote by \mathfrak{g}_α the eigenspace associated with α . Each \mathfrak{g}_α is one dimensional [11, Theorem 7.23]. We can easily show that

$$\mathfrak{b}_\mathfrak{t} = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha \tag{3}$$

is a Lie subalgebra of $\mathbb{C}\mathfrak{g}$.

Since \mathfrak{t} is abelian, any choice of vector space $\mathfrak{m} \subset \mathbb{C}\mathfrak{t}$ is a Lie algebra and $\mathfrak{h} = \mathfrak{m} \oplus \mathfrak{b}_\mathfrak{t}$ is a Lie algebra. If \mathfrak{m} is elliptic (resp. CR), then \mathfrak{h} is elliptic (resp. CR). Also, notice that $\mathfrak{b}_\mathfrak{t}$ is an ideal of \mathfrak{h} .

Let us focus on the CR case. We denote by d the dimension of \mathfrak{t} and, since the dimension of each \mathfrak{g}_α is one, we can easily see that $N = d + 2l$ with l being the number of elements of Δ_+ . If \mathfrak{m} is a Lie subalgebra of $\mathbb{C}\mathfrak{t}$ of dimension $[d/2]^1$ such that its intersection with \mathfrak{g} is null; the sum $\Phi(\mathfrak{m})$ of \mathfrak{m} and $\mathfrak{b}_\mathfrak{t}$ is a subalgebra of dimension $[N/2]$ and null intersection with \mathfrak{g} .

We recall the two subalgebras \mathfrak{h}_1 and \mathfrak{h}_2 of \mathfrak{g} are called conjugate if there exist a $g \in G$ such that $Ad(g)\mathfrak{h}_1 = \mathfrak{h}_2$. Here $Ad(g)$ is the differential at $e \in G$ of the map $C_g : G \rightarrow G$ given by $C_g(h) = ghg^{-1}$.

¹ For $x \in \mathbb{R}$, we define $[x] = \max\{n \in \mathbb{Z} : n \leq x\}$.

Definition 1 We shall say that a subalgebra $\mathfrak{h} \subset \mathbb{C}\mathfrak{g}$ is of type CR0 if it is conjugate to a subalgebra of the form

$$\Phi(\mathfrak{m}) = \mathfrak{m} \oplus \mathfrak{b}_t \tag{4}$$

in which \mathfrak{m} is a subspace of dimension $[d/2]$ of $\mathbb{C}\mathfrak{t}$ and null intersection with \mathfrak{g} .

When N is odd, we introduce the set $\mathcal{D}(\Delta_+)$ of all elements of the form $(\alpha, \mathfrak{m}, x, t)$ with α a simple root in Δ_+ , \mathfrak{m} a subspace of dimension $[d/2]$ of the kernel of α with null intersection with \mathfrak{g} such that x is an element not null of \mathfrak{g}_α and $t \in \mathfrak{t}$. For every $(\alpha, \mathfrak{m}, x, t)$ in $\mathcal{D}(\Delta_+)$, we define

$$\Theta(\alpha, \mathfrak{m}, x, t) = \mathfrak{m} \oplus \bigoplus_{\beta \in \Delta_+ \setminus \{\alpha\}} \mathfrak{g}_\beta \oplus \text{span}_{\mathbb{C}}\{t + x\}. \tag{5}$$

Therefore, $\Theta(\alpha, \mathfrak{m}, x, t)$ is a subalgebra of $\mathbb{C}\mathfrak{g}$ of dimension $[N/2]$ and null intersection with \mathfrak{g} .

Definition 2 We shall say that a subalgebra $\mathfrak{h} \subset \mathbb{C}\mathfrak{g}$ is of type CR1 if it is conjugate to a subalgebra of the form $\Theta(\alpha, \mathfrak{m}, x, t)$ in which $(\alpha, \mathfrak{m}, x, t)$ is an element of $\mathcal{D}(\Delta_+)$.

Without loss of generality, we take \mathfrak{h} to be of the form (4) or (5) instead of conjugate to this form.

For an algebra \mathfrak{h} of type CR0 or CR1, we call the subalgebra \mathfrak{m} the toric part of \mathfrak{h} .

In Sect. 5, simple examples of both types of CR structures are given for the group $SU(2)$.

The main theorem of [6] is the following:

Theorem 4 *Let \mathcal{V} be a left-invariant CR structure of maximum rank over a Lie group G . Then, the rank of \mathcal{V} is $[N/2]$. If N is even, then the corresponding subalgebra \mathfrak{h} given by the set of all left-invariant vector fields of \mathcal{V} is a complex structure and it is of type CR0. If N is odd, the corresponding subalgebra \mathfrak{h} is of type CR0 or CR1.*

Theorem 5 *A semisimple Lie group G does not admit a Levi-flat CR structure of type CR0.*

Proof Recall that we limit ourselves to CR structures of maximal rank, i.e., the codimension of $\mathcal{V} \oplus \bar{\mathcal{V}}$ is one. Thus, we only consider odd dimensional Lie groups.

Let \mathfrak{g} be the complexified Lie algebra of the group. If \mathfrak{g} is not semisimple, then G is not semisimple. And to show \mathfrak{g} is not semisimple it suffices to show that $[\mathfrak{g}, \mathfrak{g}]$ is a proper subset of \mathfrak{g} . A CR structure of type CR0 has the form

$$\mathcal{V} = \mathfrak{m} \oplus \bigoplus_{\beta \in \Delta^+} \mathfrak{g}_\beta$$

where $\mathfrak{m} \oplus \bar{\mathfrak{m}}$ is of codimension one in some maximal toral subalgebra of \mathfrak{g} . Call this toral subalgebra \mathfrak{t} . Note that there is an element T in the real Lie algebra of G with

$$\mathfrak{t} = \{T\} \oplus \mathfrak{m} \oplus \bar{\mathfrak{m}}$$

We have the corresponding decompositions

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\beta \in \Delta} \mathfrak{g}_\beta = \{T\} \oplus \mathcal{V} \oplus \bar{\mathcal{V}}.$$

From

$$[T, \mathfrak{m}] = 0$$

and

$$[T, \mathfrak{g}_\beta] \subset \mathfrak{g}_\beta$$

we see that

$$[T, \mathcal{V}] \subset \mathcal{V} \text{ and } [T, \bar{\mathcal{V}}] \subset \bar{\mathcal{V}}.$$

The involutivity and Levi flatness conditions give us

$$[\mathcal{V} \oplus \bar{\mathcal{V}}, \mathcal{V} \oplus \bar{\mathcal{V}}] \subset \mathcal{V} \oplus \bar{\mathcal{V}}$$

and so we now have

$$[\mathfrak{g}, \mathfrak{g}] \subset \mathcal{V} \oplus \bar{\mathcal{V}}$$

which is a proper subset of \mathfrak{g} . □

Note that this proof doesn't apply to structures of type CR1. In place of $[T, \mathcal{V}] \subset \mathcal{V}$, we only obtain

$$[T, \mathcal{V}] \subset \mathcal{V} \oplus \mathfrak{g}_\alpha$$

(in the notation of Definition 2), from which we cannot conclude

$$[T, \mathcal{V} \oplus \bar{\mathcal{V}}] \subset \mathcal{V} \oplus \bar{\mathcal{V}}.$$

The following result was communicated to the authors by a referee.

Theorem 6 *A compact Lie group G does not admit a Levi-flat CR structure of type CR1.*

Proof Let G be a compact Lie group with Lie algebra \mathfrak{g} and let $\mathfrak{h} \subset \mathbb{C}\mathfrak{g}$ be a Levi-flat structure of hypersurface type. Since \mathfrak{h} is Levi-flat, we have that $\mathfrak{h} + \bar{\mathfrak{h}}$ is a subalgebra of $\mathbb{C}\mathfrak{g}$. Let $\mathfrak{k} \doteq (\mathfrak{h} + \bar{\mathfrak{h}}) \cap \mathfrak{g}$ and because \mathfrak{h} is of hypersurface type, we have that $\dim \mathfrak{k} = \dim \mathfrak{g} - 1$.

By the compactness of G , it follows that \mathfrak{g} admits a positive ad-invariant inner product $\langle \cdot, \cdot \rangle$ (see Proposition 4.24 of [13] and its proof) and we can consider the decomposition $\mathfrak{g} = \mathfrak{k}^\perp \oplus \mathfrak{k}$. Let $X \in \mathfrak{k}^\perp$ and $Y \in \mathfrak{k}$. By ad-invariance, we have

$$0 = \langle [X, X], Y \rangle = \langle X, [X, Y] \rangle$$

and so $[X, Y] \in \mathfrak{k} = (\mathfrak{k}^\perp)^\perp$ which means that \mathfrak{k} is an ideal in \mathfrak{g} .

Notice that $\mathbb{C}\mathfrak{k} = \mathfrak{h} \oplus \bar{\mathfrak{h}}$, and thus, \mathfrak{h} defines a complex structure on $\mathbb{C}\mathfrak{k}$. Since G is compact, we have that \mathfrak{g} is reductive, and since \mathfrak{k} is an ideal in \mathfrak{g} , it follows that \mathfrak{k} is reductive. Now the result follows from [15, Section 3, Theorem 1]. □

The compactness is crucial, as can be seen in the following.

Example 2 Consider the usual basis of \mathfrak{sl}_2

$$T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with bracket products

$$[T, X] = 2X, \quad [T, Y] = -2Y, \quad [X, Y] = T.$$

Thus,

$$\mathbb{C} \otimes \mathfrak{sl}_2 = \mathfrak{M} \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_{-2}$$

with

$$\mathfrak{m} = \{T\}, \quad \mathfrak{g}_2 = \{X\}, \quad \mathfrak{g}_{-2} = \{Y\}.$$

Take $L = X + iT$ and note that L is of type CR1. Also note that

$$[L, \bar{L}] = 2i(L + \bar{L}).$$

So the semi-simple group SL_2 admits a Levi-flat CR structure (of type CR1).

This example is taken from Section 4 of [4]. It follows from the methods of that section that up to CR equivalence this is the only left-invariant Levi-flat CR structure on SL_2 .

4 Leray–Hirsch theorem for CR bundles

Let G be an odd-dimensional compact Lie group and let \mathfrak{h} be a Lie algebra in the form (4). We define $T = \exp_G(\mathfrak{t})$ and thus T is maximal torus endowed with a maximal CR structure \mathfrak{m} . We have that the inclusion $i : T \subset G$ is a map compatible with the involutive structures on T and on G , meaning that $i_*(\mathfrak{m}) \subset \mathfrak{h}$. We also have that the quotient map $\pi : G \rightarrow \Omega \doteq G/T$ induces a complex structure on Ω , that is, Ω is endowed with the involutive structure $\pi_*(\mathfrak{h})$ (see [9, Problem 2.57] for details).

In the following, we recall the definition of principal bundle, and we prove that $T \xrightarrow{i} G \xrightarrow{\pi} \Omega$ is a smooth principal bundle. We also prove that if \mathfrak{h} is a Levi-flat CR subalgebra, then the local smooth trivialization for the bundle G is compatible with the involutive structure on the fiber and on the base space.

Definition 3 A *principal fiber bundle* P with structure group H is a triple (P, H, R) with P being a manifold, H a Lie group and $R : P \times H \rightarrow P$ a smooth right action of H on P satisfying:

1. $\Omega \doteq P/H$ has a manifold structure making the projection $\pi : P \rightarrow \Omega$ smooth;
2. P is locally trivial, i.e., there is an open cover $\{U_j\}$ of Ω and diffeomorphisms $\Phi_j : \pi^{-1}(U_j) \rightarrow U_j \times H$ satisfying

$$\Phi_j^{-1}(x, hg) = \Phi_j^{-1}(x, h)g$$

for all $x \in U_j$ and all $g, h \in H$.

Example 3 Let G be a compact Lie group and let $K \subset G$ be a closed subgroup. Then, G can be regarded as a principal bundle with structure group K endowed with a right action $R : G \times K \rightarrow G$. The quotient by this action defines a manifold $\Omega = \{gK : g \in G\}$ with the projection $\pi : G \rightarrow \Omega$ being smooth.

By [12, Corollary 10.1.11], it is possible to find a covering $\{U_j\}$ of Ω and smooth sections σ_j of the quotient map $\pi : G \rightarrow \Omega$ such that

$$(u, t) \in U_j \times K \mapsto \Psi_j(u, t) \doteq \sigma_j(u)t \in \sigma_j(U_j)K \tag{6}$$

is a diffeomorphism onto an open subset of G which we denote by V_j .

The local trivialization follows from the fact that Ω has an open cover $\{U_j\}$ and local sections $\sigma_j : U_j \rightarrow G$ for the projection π such that $\Psi_j : (u, k) \in U_j \times K \mapsto \sigma_j(u)k \in \pi^{-1}(U_j)$ is a diffeomorphism. We define $\Phi_j = \Psi_j^{-1}$ and notice that

$$\Phi_j^{-1}(x, hg) = \Psi_j(x, hg) = \sigma_j(x)hg = \Psi_j(x, h)g = \Phi_j^{-1}(x, h)g.$$

We define the analogous principal fiber bundle for CR structures.

Definition 4 A CR principal bundle P with structure group H is a triple (P, H, R) with P being a CR manifold, H a Lie group and $R : P \times H \rightarrow P$ a free CR right action of H on P satisfying:

1. $\Omega \doteq P/H$ has a CR structure making the projection $\pi : P \rightarrow \Omega$ a CR map;
2. P is locally CR trivial, i.e., there is an open cover $\{U_j\}$ of Ω and CR diffeomorphisms $\Phi_j : \pi^{-1}(U_j) \rightarrow U_j \times H$ satisfying

$$\Phi_j^{-1}(x, hg) = \Phi_j^{-1}(x, h)g$$

for all $x \in U$ and all $g, h \in H$.

Proposition 1 Let G be a compact Lie group endowed with a left-invariant Levi-flat CR structure \mathfrak{h} of maximal rank and let $\mathfrak{m} \subset \mathfrak{h}$ be its toric part. Let \mathfrak{k} be the Lie algebra of the maximal torus T such that $\mathfrak{m} \subset \mathbb{C}\mathfrak{k}$. Then, G is CR principal bundle with structure group T and base space $\Omega = G/T$.

In order to prove Proposition 1, we need: a few results:

Lemma 1 Let G be a compact Lie group endowed with a left-invariant CR structure \mathfrak{h} of maximal rank and let $\mathfrak{m} \subset \mathfrak{h}$ be its toric part and let T be the associated maximal torus. Let $\hat{G} \times T$ be endowed with the CR structure given by $\mathfrak{h} \oplus \mathfrak{m}$. Then $\mu : (g, t) \in \hat{G} \times T \mapsto gt \in G$ is a CR map.

Proof Let $X \oplus Y \in \mathfrak{h} \oplus \mathfrak{m}$ and notice that

$$(\mu_*)_{(a,b)}(X \oplus Y) = (R_b)_*(X) + (L_a)_*(Y).$$

Let \mathfrak{h}_g denote $(L_g)_*\mathfrak{h}$, so $(L_a)_*(Y) \in \mathfrak{h}_{ab}$; hence, we only need to verify that $(R_b)_*(X) \in \mathfrak{h}_{ab}$.

We write $X = X' + \sum_{\alpha \in \Delta_+} X_\alpha$ with $X' \in \mathfrak{m}$ and $X_\alpha \in \mathfrak{g}_\alpha$. Clearly, we have that $(R_b)_*(X') \in \mathfrak{h}_{ab}$, and by linearity, we just need to know what happens with $(R_b)_*(X_\alpha)$. Let $W \in \mathbb{C}T_bT$ and recall that $[W, X_\alpha] = \alpha(W)X_\alpha$. Also, notice that

$$[W_{ab}, (R_b)_*(X_\alpha)] = [(R_b)_*(W_a), (R_b)_*(X_\alpha)] = (R_b)_*[W_a, X_\alpha] = (R_b)_*\alpha(W)X_\alpha.$$

Therefore, $(R_b)_*(X_\alpha) \in \text{Ker}(\text{ad}_W - \alpha(W)\text{id}) = \mathfrak{g}$, with $\text{ad}_W(X) = [W, X]$. Now, since $\dim_{\mathbb{C}} \mathfrak{g}_\alpha = 1$, we have that there exists a $\lambda_{b,\alpha} \in \mathbb{C}$ such that

$$(R_b)_*(X_\alpha) = \lambda_{b,\alpha} X_\alpha.$$

□

Lemma 2 Let M be a smooth manifold endowed with a complex or a Levi-flat CR structure \mathcal{V} and N be a smooth manifold endowed with a complex structure \mathcal{S} . Let $f : M \rightarrow N$ be a map such that $f_*(\mathcal{V}_x) = \mathcal{S}_{f(x)}$ for all $x \in M$. Then, for every $y \in N$ there exist an open neighborhood U of y and a map $\sigma : U \rightarrow M$ such that $f \circ \sigma(x) = x$ for all $x \in U$ and $\sigma_*(\mathcal{S}_x) \subset \mathcal{V}_\sigma(x)$ for all $x \in U$.

Proof Let $y \in N, x \in f^{-1}(\{y\})$ and $V \subset M$ be an open neighborhood of x with coordinates $\phi : V \rightarrow V' \times V'' \subset \mathbb{C}^\nu \times \mathbb{R}^{n-\nu}$ with $V' \subset \mathbb{C}^\nu$ and $V'' \subset \mathbb{R}^{n-\nu}$. When \mathcal{V} is complex, we take $\nu = n$. We choose the coordinates

$$\phi = (z_1, \dots, z_\nu, t_1, \dots, t_{n-\nu})$$

such that over V the involutive structure \mathcal{V} is generated by the vector fields

$$\frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_v}$$

Let $U \subset N$ be a neighborhood of y with coordinates $\psi : U \rightarrow U' \subset \mathbb{C}^l$ written as

$$\psi = (w_1, \dots, w_l)$$

such that over U the involutive structure \mathcal{S} is generated by the vector fields

$$\frac{\partial}{\partial \bar{w}_1}, \dots, \frac{\partial}{\partial \bar{w}_l}$$

Now, by using these coordinates, we write a local representation of f , namely $\tilde{f} : V' \rightarrow U'$, as $\tilde{f} = \psi \circ f \circ \phi^{-1}$. Notice that since $f_*(\mathcal{V}_x) = \mathcal{S}_{f(x)}$, we have that

$$f_*(\mathcal{V}_x + \overline{\mathcal{V}_x}) = \mathcal{S}_{f(x)} + \overline{\mathcal{S}_{f(x)}} = \mathbb{C}T_{f(x)}N.$$

Let $J\tilde{f}(z, t)$ denote the Jacobian matrix of \tilde{f} . Since \tilde{f}_* is surjective, by shrinking the open sets V, V', U and U' if necessary, and by rearranging the indices, we can assume that first $l \times l$ block of the Jacobian matrix of \tilde{f} is invertible. We write $z' = (z_1, \dots, z_l), (z'_0, z''_0, t_0) = \phi(x), w_0 = \psi(y)$, and we define the function $F(z', z'', t) = (\tilde{f}(z', z'', t_0), z'', t)$. Therefore, from the Inverse Function Theorem there exist F^{-1} and this function is holomorphic in the z', z'' and smooth in t . Let $F'(z, t) = (F(z), t)$, notice that this function is invertible, and define $\phi' = F' \circ \phi$. Notice that with ϕ' as new coordinate on V we have that the local representation of f is the projection on z' .

Therefore, we can define a holomorphic section for f by defining $\sigma'(z', z'', t) = (z', z''_0, t_0)$ and then defining $\sigma : U \rightarrow M$ as $\sigma = \phi^{-1} \circ \tilde{\sigma} \circ \phi$. Thus, we have that $f \circ \sigma = \text{Id}_U$ and by construction, since $\tilde{\sigma}, \tilde{\phi}$ and ψ are compatible with the structures \mathcal{V} and \mathcal{S} , we have that $\sigma_*(w) \in V_{\sigma(x)}$ for all $w \in W_x$. □

Proof of Proposition 1 We apply Lemma 2 with $M = G$ and $N = \Omega (= G/T)$ and thus have for every small open set U and every $t \in T$, the CR diffeomorphism

$$\Phi : (u, t) \in U \times T \mapsto \sigma(u)t \in \sigma(U)T = \pi^{-1}(U). \tag{7}$$

Let $X \oplus Y \in \pi_*(\mathfrak{h}|_u \oplus \mathfrak{m})$. We want to prove that $(\Phi_*)_{(u,t)}(X \oplus Y) \in \mathfrak{h}_{\sigma(u)t}$. First, we compute $(\Phi_*)_{(u,t)}(0 \oplus Y)$. Let $\alpha, \beta : (-\epsilon, \epsilon) \rightarrow G$ be two smooth curves satisfying $\alpha(0) = \beta(0) = t$ and $\alpha'(0) + i\beta'(0) = Y$. Then,

$$\begin{aligned} (\Phi_*)_{(u,t)}(0 \oplus Y) &= \left. \frac{d}{ds} \right|_{s=0} \sigma(u)\alpha(s) + i \left. \frac{d}{ds} \right|_{s=0} \sigma(u)\beta(s) \\ &= (L_{\sigma(u)})_*(\alpha'(0)) + i(L_{\sigma(u)})_*(\beta'(0)) \\ &= (L_{\sigma(u)})_*(Y) \in \mathfrak{h}_{\sigma(u)t} \end{aligned}$$

Now, we compute $(\Phi_*)_{(u,t)}(X \oplus 0)$. Let $\alpha, \beta : (-\epsilon, \epsilon) \rightarrow G$ be two smooth curves satisfying $\alpha(0) = \beta(0) = u$ and $\alpha'(0) + i\beta'(0) = X$. Then,

$$\begin{aligned} (\Phi_*)_{(u,t)}(X \oplus 0) &= \left. \frac{d}{ds} \right|_{s=0} \sigma(\alpha(s))t + i \left. \frac{d}{ds} \right|_{s=0} \sigma(\beta(s))t \\ &= (R_t)_* \circ \sigma_* \alpha'(0) + i(R_t)_* \circ \sigma_* \beta'(0) \\ &= (R_t)_* \circ \sigma_*(X) \end{aligned}$$

We use that R_t and σ are CR maps, and the proof is complete. □

Remark 1 Notice that the existence of local sections for π implies that (7) is a CR diffeomorphism, and since the set $U \times T$ is endowed with a Levi-flat structure, we conclude that the existence of local sections implies that the original structure is Levi-flat. Therefore, the Levi-flat condition from Proposition 1 cannot be dropped.

In the following, we show an example of a left-invariant CR structure of type CR0 such that the map μ , as defined in Lemma 1, is a CR map, but, since such structure is not Levi-flat, it fails to be a CR principal bundle with structure group T .

Example 4 Let G be the compact Lie group

$$SU(2) \doteq \left\{ \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix} : z_1, z_2 \in \mathbb{C}, |z_1|^2 + |z_2|^2 = 1 \right\}$$

and let $\mathbb{T} \subset G$ given by

$$\mathbb{T} = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in \mathbb{R} \right\}.$$

It is easy to prove that \mathbb{T} is a maximal torus. Recall that the Lie algebra of $SU(2)$, which we denote by $\mathfrak{su}(2)$, is generated by

$$X = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The vector field $L = X \otimes 1 - Y \otimes i$ defines a left-invariant CR structure $\mathfrak{h} = \text{span}_{\mathbb{C}}\{L\} \subset \mathbb{C}\mathfrak{g}$. For $t \in \mathbb{T}$, we can easily compute $(R_t)_*(L) = (R_t)_*(X) \otimes 1 - (R_t)_*(Y) \otimes i$. In fact, we write

$$t = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

for a $t \in \mathbb{R}$ and we get

$$(R_t)_*(X) = Xt = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \begin{pmatrix} 0 & ie^{-i\theta} \\ ie^{i\theta} & 0 \end{pmatrix},$$

and

$$(R_t)_*(Y) = Yt = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \begin{pmatrix} 0 & -e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}$$

thus

$$(R_t)_*(L) = \begin{pmatrix} 0 & ie^{-i\theta} \\ ie^{i\theta} & 0 \end{pmatrix} \otimes 1 - \begin{pmatrix} 0 & -e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix} \otimes i$$

and we easily verify that $(R_t)_*(L) = e^{-i\theta}L$. Thus, μ is a CR map.

We will use the following special case of the Künneth formula.

Proposition 2 Let $D \subset \mathbb{C}^n$ be a polydisk with its natural complex structure \mathcal{V} and let U be a smooth manifold endowed with a CR structure \mathcal{W} . We assume that there are globally defined linearly independent $\bar{\partial}_b$ -closed forms ζ_1, \dots, ζ_m spanning \mathcal{W}^\perp . Then,

$$H^{0,q}(D \times U; \mathcal{V} \oplus \mathcal{W}) \cong \mathcal{O}(D) \otimes H^{0,q}(U; \mathcal{V})$$

with $\mathcal{O}(D)$ the set of all holomorphic functions on D .

Proof We denote by $\bar{\partial}$ the differential operator associated with \mathcal{V} , by $\bar{\partial}_b$ the differential operator associated with \mathcal{W} , and by d' the differential operator associated with $\mathcal{V} \oplus \mathcal{W}$. Notice that $d' = \bar{\partial} + \bar{\partial}_b$ and since $d' \circ d' = 0$ we conclude that $\bar{\partial} \circ \bar{\partial}_b + \bar{\partial}_b \circ \bar{\partial} = 0$. We denote by π_D the projection on the first variable and by π_U the projection on the second variable.

Notice that if $f \in \mathcal{O}(D)$ and $[u] \in H^{0,q}(U; \mathcal{V})$, then $\pi_D^*(f)\pi_U^*(u)$ defines an element in $H^{0,q}(D \times U; \mathcal{V} \oplus \mathcal{W})$ and since the elements of the form $\pi_D^*(f)[\pi_U^*(u)]$ form a basis for $\mathcal{O}(D) \otimes H^{0,q}(U; \mathcal{V})$, we have a homomorphism

$$\psi : \mathcal{O}(D) \otimes H^{0,q}(U; \mathcal{V}) \mapsto H^{0,q}(D \times U; \mathcal{V} \oplus \mathcal{W}).$$

We will prove that ψ is an isomorphism.

First we prove that ψ is surjective. Let $[u] \in H^{0,q}(D \times U; \mathcal{V} \oplus \mathcal{W})$. We write $u = \sum_{|J|+|K|=q} u_{JK} d\bar{z}_J \wedge \tau_K$. We can decompose u as a finite sum $u = \sum_{j+k=q} u_{jk}$ with $u_{jk} = \sum_{|J|=j, |K|=k} u_{IJ} d\bar{z}_J \wedge \tau_K$. Let u_{jk} be such that j is as big as possible with $u_{jk} \neq 0$.

From $d'u = 0$ and the maximality of j , we have that $\bar{\partial}u_{jk} = 0$. Since D is a polydisk, we can find v_{jk} in D with parameters in U such that $\bar{\partial}v_{jk} = u_{jk}$. Now we can take $u' = u - \bar{\partial}v_{jk}$. Notice that u' and u are in the same cohomology class. Now, let u'_{jk} be a nonzero form such that j is as big as possible. This j is less than the one obtained with u_{jk} . By repeating this process, after a finite number of steps we find a \tilde{u} that can be represented as $\tilde{u} = \sum_{|K|=q} \tilde{u}_K \tau_K$ with $\tilde{u}_K \in C^\infty(D \times U)$ holomorphic in D .

Now we write the Taylor series at $0 \in D$ for each \tilde{u}_K and we obtain

$$\tilde{u}_K = \sum_{\alpha \in \mathbb{Z}_+^n} \frac{\partial^\alpha u_K}{\partial z^\alpha}(0, t) \frac{z^\alpha}{\alpha!}, \quad z \in D, t \in U$$

with uniform convergence over compact sets. Notice that

$$\tilde{u} = \sum_{\alpha \in \mathbb{Z}_+^n} \frac{z^\alpha}{\alpha!} \left(\sum_{|K|=q} \frac{\partial^\alpha u_K}{\partial z^\alpha}(0, t) \tau_K \right)$$

and each term $\sum_{|K|=q} \frac{\partial^\alpha u_K}{\partial z^\alpha}(0, t) \tau_K$ defines a cohomology class in $H^{0,q}(U; \mathcal{V})$. This proves surjectivity.

Now assume that $[u] \in \mathcal{O}(D) \otimes H^{0,q}(U; \mathcal{V})$ is such that $\psi([u]) = 0$. This means that there exists v such that $d'v = u$. We write $v = \sum_{|J|+|K|=q-1} v_{JK} d\bar{z}_J \wedge \tau_K$, and we see that $\bar{\partial}v_{JK} = 0$ if $|J| \geq 1$ and with a process similar to the one we did in the surjectivity case conclude that we have a solution \tilde{v} with $\bar{\partial}\tilde{v} = u$ with $\tilde{v} \in \mathcal{O}(D) \otimes H^{0,q-1}(U; \mathcal{V})$. \square

Let (P, G, R) be a principal CR bundle with base space B . Let \mathcal{U}, \mathcal{V} and \mathcal{W} be the CR structures of, respectively, G, P and B and denote by $r_x : H^{0,q}(P; \mathcal{V}) \rightarrow H^{0,q}(P_x; \mathcal{V}|_{P_x})$ the restriction to the fiber P_x . A homomorphism $e : H^{0,q}(G; \mathcal{U}) \rightarrow H^{0,q}(P; \mathcal{V})$ is called a *cohomology extension of the fiber* if for every $x \in P$ the composition $H^{0,q}(G; \mathcal{U}) \xrightarrow{e} H^{0,q}(P; \mathcal{V}) \xrightarrow{r_x} H^{0,q}(P_x; \mathcal{V}|_{P_x})$ is an isomorphism. We also assume that if $V \subset P$ is a small neighborhood such that there exist a trivialization $\Psi : V \rightarrow U \times G$ with $U = \pi(V)$ and if pr_G is the projection $\text{pr}_G : U \times G \rightarrow G$, then

$$\Psi^* \circ \text{pr}_G^*(u) = e(u) \tag{8}$$

for all $u \in H^{0,q}(G; \mathcal{U})$ and any $q = 0, 1, 2, \dots$

Theorem 7 (Leray–Hirsch theorem for CR principal bundles) *Let (P, H, R) be a principal CR bundle. Let \mathcal{V} be the CR structure of P and assume that there are globally defined linearly*

independent $\bar{\partial}_b$ -closed forms ζ_1, \dots, ζ_m spanning \mathcal{V}^\perp . Suppose that each $H^{0,q}(H)$ is finite-dimensional and that there is a cohomology extension $e : H^{0,q}(H) \rightarrow H^{0,q}(P)$, then there is an isomorphism

$$H^{0,q}(P; \mathcal{V}) \cong \sum_{r+s=q} H^{0,s}(H; \mathcal{U}) \otimes H^{0,r}(\Omega; \mathcal{W}),$$

with \mathcal{U} the CR structure of H and \mathcal{W} the CR structure of the base space Ω .

Proof Let $\{U_j\}$ be an open covering of Ω by sets such that each U_j is biholomorphic to a polydisk, and there exists a local CR trivialization $\Phi_j : V_j \rightarrow U_j \times \Omega$ of Ω with $V_j = \pi^{-1}(U_j)$. Notice that $\{V_j\}$ is an open covering for P . Now we have an isomorphism

$$H^{0,q}(V_j; \mathcal{V}) = H^{0,q}(U_j \times H; \mathcal{S} \oplus \mathcal{U})$$

and we have a homomorphism

$$\psi : \sum_{r+s=q} H^{0,r}(U_j; \mathcal{W}) \otimes H^{0,s}(H; \mathcal{U}) \rightarrow H^{0,q}(U_j \times H; \mathcal{S} \oplus \mathcal{U})$$

given by $[u] \otimes [v] \mapsto \pi_{U_j}^*(u) \wedge \pi_H^*(v)$. By Lemma 2, this last homomorphism is an isomorphism. These two isomorphisms, combined with the fact that e is a cohomology extension, imply that we have for each V_j an isomorphism

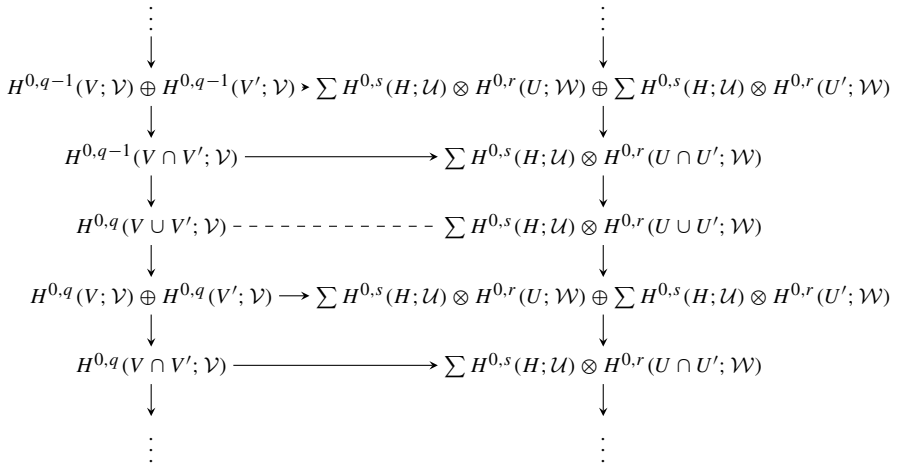
$$\phi : \sum_{r+s=q} H^{0,r}(U_j; \mathcal{W}) \otimes H^{0,s}(H; \mathcal{U}) \rightarrow H^{0,q}(U_j \times H; \mathcal{S} \oplus \mathcal{U})$$

given by $[u] \otimes [v] \mapsto \pi^*(u) \wedge e(v)$.

On the other hand, for any two sets $U, U' \in \{U_j\}$ we can use Mayer–Vietoris sequence to get an exact sequence

$$\begin{aligned} \dots &\rightarrow H^{0,r}(U \cup U'; \mathcal{W}) \rightarrow H^{0,r}(U; \mathcal{W}) \oplus H^{0,r}(U'; \mathcal{W}) \rightarrow H^{0,r}(U \cap U'; \mathcal{W}) \\ &\rightarrow H^{0,r+1}(U \cup U'; \mathcal{W}) \rightarrow \dots \end{aligned}$$

Now, we tensor each term with the finite vector space $H^{0,s}(H; \mathcal{U})$ and we sum all terms satisfying $r + s = q$ and we obtain another exact sequence. We define $V_j = \pi^{-1}(U_j)$. We take $V = \pi^{-1}(U)$ and $V' = \pi^{-1}(U')$. Now we have two sequences, which we show side by side emphasizing the isomorphism between each element. Notice that the only missing isomorphism is the term in the middle.



Now, since this diagram is commutative, we can apply the Five Lemma and construct the missing isomorphism. Since Ω is compact, it has a finite covering; thus, we can use an induction on the covering and construct an isomorphism defined in the union of all $\{U_j\}$ and $\{V_j\}$. That is, we obtain

$$H^{0,q}(P; \mathcal{V}) = \sum_{r+s=1} H^{0,s}(H; \mathcal{U}) \otimes H^{0,r}(\Omega; \mathcal{W}).$$

□

Lemma 3 *Let G be a compact Lie group endowed with a left-invariant involutive structure \mathfrak{h} of the form*

$$\mathfrak{h} = \mathfrak{m} \oplus \mathfrak{u}$$

with $\mathfrak{m} \subset \mathbb{C}\mathfrak{t}$ and \mathfrak{t} the Lie algebra of a maximal torus T and \mathfrak{u} an ideal of \mathfrak{h} . Let $u \in C^\infty(T; \Lambda^{0,q})$ be a left-invariant and $\bar{\partial}'_b$ -closed, with $\bar{\partial}'_b$ being the differential operator associated with \mathfrak{m} on T . Then, u can be extended to a $\bar{\partial}_b$ -closed form in G which restricts to a left-invariant form on each leaf gT .

Proof Since u is left-invariant, we can regard it as an element of the dual of \mathfrak{m} and we can extend u as an element of $\mathbb{C}\mathfrak{g}^*$ by defining it as zero if any of its arguments are in \mathfrak{u} or $\bar{\mathfrak{u}}$. We denote $u + \bar{u}$ as \mathfrak{m}^\perp . Let $w_g = (L_{g^{-1}})^*u$ and notice that

$$(L_g)^* : H^{0,q}(gT; \mathfrak{m}) \rightarrow H^{0,q}(T; \mathfrak{m})$$

is an isomorphism and so w is a smooth $(0, q)$ -form in G which restricts to a cohomology class in each leaf gT .

We are going to see that w is also $\bar{\partial}'_b$ -closed. We just need to show that w satisfies $dw(X_1, \dots, X_{q+1}) = 0$ if all the arguments are in \mathfrak{h} . Let $X_1, \dots, X_{q+1} \in \mathfrak{h}$. Since the exterior derivative commutes with the pullback, we have

$$dw_g = d[(L_{g^{-1}})^*u] = (L_{g^{-1}})^*(du)$$

and we just need to do the following computation

$$\begin{aligned} du(X_1, \dots, X_{q+1}) &= \sum_{j=1}^{q+1} (-1)^{j+1} X_j u(X_1, \dots, \hat{X}_j, \dots, X_{q+1}) \\ &\quad + \sum_{j < k} (-1)^{j+k+1} u([X_j, X_k], X_1, \dots, \hat{X}_j, \dots, \hat{X}_k, \dots, X_{q+1}) \\ &= \sum_{j < k} (-1)^{j+k+1} u([X_j, X_k], X_1, \dots, \hat{X}_j, \dots, \hat{X}_k, \dots, X_{q+1}). \end{aligned}$$

Notice that by left-invariance, we have that $X_j u(X_1, \dots, \hat{X}_j, \dots, X_{q+1})$ is zero. If we assume that $X_j \in \mathfrak{m}$ for all j , then $(dw_g)(X_1, \dots, X_{q+1}) = 0$ because $\overline{\partial}'_b u = 0$ and if we assume that $X_1 \in \mathfrak{u}$ and that $X_j \in \mathfrak{u}$ for some $j > 1$. Then,

$$\begin{aligned} &\sum_{j < k} (-1)^{j+k+1} u([X_j, X_k], X_1, \dots, \hat{X}_j, \dots, \hat{X}_k, \dots, X_{q+1}) \\ &= \sum_{1 < k} (-1)^{1+k+1} u([X_1, X_k], X_2, \dots, \hat{X}_j, \dots, \hat{X}_k, \dots, X_{q+1}) = 0 \end{aligned}$$

because \mathfrak{u} is an ideal and so $[X_1, X_k] \in \mathfrak{u}$.

If we assume that two or more elements are $X_1, X_2 \in \mathfrak{m}^\perp$, we use the fact that \mathfrak{u} is an ideal and an argument similar to the one above proves that we have

$$(dw_g)(X_1(g), \dots, X_{q+1}(g)) = 0.$$

□

Remark 2 Notice that if \mathfrak{h} is a CR algebra of type CR0, then we can use Lemma 3 with $\mathfrak{u} = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$. We refer to [6] for a proof that, in both cases, \mathfrak{u} is an ideal of \mathfrak{h} .

Lemma 4 Let G, T, \mathfrak{h} and \mathfrak{m} be as in Lemma 3 and assume that every cohomology class of $H^{0,q}(T; \mathfrak{m})$ admits a left-invariant representative. Then, the cohomology extension

$$e : H^{0,q}(T; \mathfrak{m}) \hookrightarrow H^{0,q}(G; \mathfrak{h})$$

satisfies

$$\text{pr}_T^*(u) = \Phi_U^*(e(u))$$

for all cohomology classes $[u] \in H^{0,q}(T; \mathfrak{m})$ with $\Phi_U : (u, t) \in U \times T \mapsto \sigma(u)t \in \sigma(U)T$ being the local trivialization map and pr_T being the projection of $U \times T$ onto T .

Proof Let $[u]$ be a cohomology class in $H^{0,q}(T; \mathfrak{m})$ with u left-invariant. Let $X_1 \oplus Y_1, \dots, X_q \oplus Y_q \in \pi_*(\mathfrak{h}) \oplus \mathfrak{m}$.

On the left hand side, we have

$$\text{pr}_T^*(u)(X_1 \oplus Y_1, \dots, X_q \oplus Y_q) = u((\text{pr}_T)_*(X_1 \oplus Y_1), \dots, (\text{pr}_T)_*(X_q \oplus Y_q)) = u(Y_1, \dots, Y_q).$$

And, on the right hand side, we have

$$\Phi_U^*(e(u))(X_1 \oplus Y_1, \dots, X_q \oplus Y_q) = e(u)((\Phi_U)_*(X_1 \oplus Y_1), \dots, (\Phi_U)_*(X_q \oplus Y_q)).$$

Notice that $(\Phi_U)_*(X_j \oplus Y_j) = (R_t)_* \circ \sigma_*(X_j) + (L_{\sigma(u)})_*(Y_j)$ (see proof of Lemma 1), and if we prove that $(R_t)_* \circ \sigma_*(X_j) \in \mathfrak{m}^\perp$, then $e(u)((\Phi_U)_*(X_1 \oplus Y_1), \dots, (\Phi_U)_*(X_q \oplus Y_q)) = 0$.

$0) = (\Phi_U)_*(X_q \oplus Y_q) = 0$ and by linearity $e(u)((\Phi_U)_*(X_1 \oplus Y_1), \dots, (\Phi_U)_*(X_q \oplus Y_q)) = u(Y_1, \dots, Y_q)$.

Let $L_j \in \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ be such that $\pi_*(L_j) = X_j$. We can write $L_j = \sum_{\alpha \in \Delta_+} L_{j,\alpha}$ with $L_{j,\alpha} \in \mathfrak{g}_\alpha$ and $X_j = \sum_{\alpha \in \Delta_+} X_{j,\alpha}$ with $X_{j,\alpha} = \pi_*(L_{j,\alpha})$.

Notice that $\pi_*((R_t)_* \circ \sigma_*(X_{j,\alpha}) - L_{j,\alpha}) = \pi_*(\sigma_*(X_{j,\alpha})) - \pi_*(L_{j,\alpha}) = 0$ and thus

$$\pi_*((R_t)_* \circ \sigma_*(X_{j,\alpha}) - L_{j,\alpha}) \in \mathbb{C}T_{\sigma(u)t}(\sigma(u)T) = \mathbb{C}\mathfrak{t}_{\sigma(u)t}.$$

Now, for any $W \in \mathbb{C}\mathfrak{t}_{\sigma(u)t}$, we have

$$0 = [W, \pi_*((R_t)_* \circ \sigma_*(X_{j,\alpha}) - L_{j,\alpha})] = [W, \pi_*((R_t)_* \circ \sigma_*(X_{j,\alpha}))] - [W, L_{j,\alpha}]$$

and so $[W, \pi_*((R_t)_* \circ \sigma_*(X_{j,\alpha}))] = \alpha(W)L_{j,\alpha}$ for all W . This means that $\pi_*((R_t)_* \circ \sigma_*(X_{j,\alpha})) \in \mathfrak{g}_\alpha$ and thus $(R_t)_* \circ \sigma_*(X_j) \in \bigoplus_{\Delta_+} \mathfrak{g}_\alpha$. □

Now we have all we need to prove Theorem 2. Let G be a connected and compact Lie group endowed with a left-invariant Levi-flat CR structure \mathcal{V} which we represent by the Lie algebra \mathfrak{h} . We are assuming that for every degree q and every cohomology class u in $H^{0,q}(T; \mathfrak{m})$ there exist a cohomology extension u' in $H^{0,q}(G; \mathcal{V})$ such that, when restricted to gT , it defines a cohomology class $H^{0,q}(gT; g\mathcal{W})$.

By Theorem 7, we have

$$H^{0,q}(G; \mathcal{V}) = \sum_{r+s=q} H^{0,s}(T; \mathcal{W}) \otimes H^{0,r}(\Omega; U).$$

Notice that, from Chapter 8 of [3], $\Omega = G/T$ has positive first Chern class, so we can apply Corollary 11.25 of [3] to obtain that $H^{0,r}(\Omega; \mathcal{W}) = 0$ for all $r > 0$. We conclude that

$$H^{0,q}(G; \mathcal{V}) = H^{0,q}(T; \mathcal{W})$$

which proves Theorem 2.

Let G be a compact Lie group with Lie algebra \mathfrak{g} and let $\mathfrak{h} \subset \mathbb{C}\mathfrak{g}$ be a Levi-flat CR algebra of maximal rank. That is, we are assuming that $2 \dim_{\mathbb{C}} \mathfrak{h} + 1 = \dim_{\mathbb{C}} \mathfrak{g}$ and since \mathfrak{h} is Levi-flat, we have that $\mathfrak{h} + \bar{\mathfrak{h}}$ is a Lie algebra so $\mathfrak{k} = (\mathfrak{h} + \bar{\mathfrak{h}}) \cap \mathfrak{g}$ is a real Lie algebra. We can easily verify that $\mathfrak{h} + \bar{\mathfrak{h}} = \mathbb{C}\mathfrak{k}$ and the group $K = \exp_G(\mathfrak{k})$ has a complex structure induced by \mathfrak{h} .

Proposition 3 *Let G be a compact Lie group with Lie algebra \mathfrak{g} . Let $\mathfrak{h} \subset \mathbb{C}\mathfrak{g}$ be a Levi-flat CR algebra of maximal rank and let $\mathfrak{k} = (\mathfrak{h} + \bar{\mathfrak{h}}) \cap \mathfrak{g}$ and suppose that $K = \exp_G(\mathfrak{k})$ is closed. Then,*

$$H^{0,q}(G; \mathfrak{h}) = H^{0,q}(K; \mathfrak{h}) \otimes C^\infty(G/K).$$

Proof The proof can be obtained by adapting the ideas from Theorem 1. □

Remark 3 Let G, K and \mathfrak{h} be as in the proposition above. Notice that \mathfrak{h} is a left-invariant complex structure and so it is of type CR0; therefore, we can decompose \mathfrak{h} as $\mathfrak{h} = \mathfrak{m} \oplus b_{\mathfrak{t}}$ with $\mathfrak{t} \subset \mathfrak{k}$ a maximal abelian subalgebra defining a maximal torus T . We have that

$$H^{0,q}(K; \mathfrak{h}) = H^{0,q}(T; \mathfrak{m})$$

and so

$$H^{0,q}(G; \mathfrak{h}) = H^{0,q}(K; \mathfrak{h}) \otimes C^\infty(G/K) = H^{0,q}(T; \mathfrak{m}) \otimes C^\infty(G/K).$$

5 Examples

Recall that $SU(2)$ is the group of all 2×2 unitary matrices having determinant 1. The Lie algebra of $SU(2)$ is denoted by $\mathfrak{su}(2)$ and is the set of all skew-Hermitian and traceless 2×2 matrices. Let $\mathfrak{k} \subset \mathfrak{su}(2)$ be any one-dimensional subalgebra. Since every connected subgroup of $SU(2)$ is closed and the rank of $SU(2)$ is 1, we have that $K = \exp_{SU(2)}(\mathfrak{k})$ is a maximal torus. Therefore, we can decompose $\mathbb{C}\mathfrak{su}(2)$ by using the roots associated with $\mathbb{C}\mathfrak{k}$. That is, we have

$$\mathbb{C}\mathfrak{su}(2) = \mathbb{C}\mathfrak{k} \oplus \mathfrak{su}(2)_\alpha \oplus \mathfrak{su}(2)_{-\alpha}.$$

Example 5 Let $X \in \mathfrak{k}$ be any element and let $Z \in \mathfrak{su}(2)_\beta$ be nonzero. We define $\mathfrak{h} = \text{span}_{\mathbb{C}}\{X + Z\}$. If $X = 0$, the subalgebra $\mathfrak{h} = \mathfrak{su}(2)_\beta$ is of type CR0 and if $X \neq 0$, \mathfrak{h} is of type CR1. These examples aren't Levi-flat.

The following a family of Levi-flat CR structures satisfying **(DC)**.

Example 6 Let $\mathbb{T} = \mathbb{T}^3$ be the 3-torus with Lie algebra \mathfrak{t} , with coordinates (x, y, t) , and endowed with the CR structure $\mathfrak{v} = \text{span}_{\mathbb{C}}\{L\}$ with $L = \partial_x + \lambda\partial_y + i\partial_t$ and $\lambda \in \mathbb{R}$. We have that $\{L, \bar{L}, \partial_y\}$ is a basis for $\mathbb{C}\mathfrak{t}$. By taking $v = (1/2)(dx - i dt)$ and $w = -\lambda dx + dy$, we have a dual basis $\{v, \bar{v}, w\}$ for $\mathbb{C}\mathfrak{t}^*$. In this case, we have that $\Lambda^{0,1} = \text{span}_{\mathbb{C}}\{dv\}$, $C^\infty(\mathbb{T}; \Lambda^{0,1}) \cong \{fv\}$ and $\bar{\partial}_b f = (Lf)u$ for all $f \in C^\infty(\mathbb{T})$. Notice that if $\lambda \in \mathbb{R} \setminus \mathbb{Q}$, then $Lf = 0$, by means of Fourier series, implies that f is a constant, this means that $H^{0,0}(\mathbb{T}; \mathfrak{v}) \cong \mathbb{C}$ and if λ in addition is a non-Liouville number, then $H^{0,1}(\mathbb{T}; \mathfrak{v}) \cong \mathbb{C}$.

6 Invariant CR structures of maximal rank on the torus

Let \mathbb{T} be a torus of dimension $N = 2n + 1$. Assume it is endowed with a left-invariant CR structure $\mathfrak{m} \subset \mathbb{C}\mathfrak{t}$ of rank n . Since \mathbb{T} is abelian, the structure \mathfrak{m} is bi-invariant. We choose a basis $\{L_1, \dots, L_n\}$ for \mathfrak{m} which we complete to a basis for $\mathbb{C}\mathfrak{t}$ denoted by $\{L_1, \dots, L_n, L_{n+1}, \dots, L_{2n+1}\}$. Notice that we can take $L_{n+j} = \bar{L}_j$ for $j = 1, \dots, n$ and take L_{2n+1} to be real.

We denote by $\{\tau_1, \dots, \tau_{2n+1}\}$ the basis dual to $\{L_1, \dots, L_n, L_{n+1}, \dots, L_{2n+1}\}$. For a ordered multi-index I , we write $L_I = (L_{i_1}, \dots, L_{i_q})$ and $\tau_I = \tau_{i_1} \wedge \dots \wedge \tau_{i_q}$.

With the notation we just described, each $u \in C^\infty(\mathbb{T}, \Lambda^{0,q})$ can be written as

$$u = \sum_{|I|=q} u_I \tau_I$$

and we have an expression for $\bar{\partial}_b$ acting on u :

$$\bar{\partial}_b u = \sum_{k=1}^n \sum_{|I|=q} L_k u_I \tau_k \wedge \tau_I$$

We want to prove that under certain conditions on the CR structure \mathfrak{m} , for each cohomology class $[u] \in H_C^{0,q}(\mathbb{T}; \mathfrak{m})$ there exist a bi-invariant form u_0 such that $u - u_0 = \bar{\partial}_b v$ for some $v \in C^\infty(\mathbb{T}; \Lambda^{0,q-1})$, that is, we have that $[u] = [u_0]$ in $H^{0,q}(\mathbb{T}; \mathfrak{m})$.

Since we are assuming that each L_j is bi-invariant, we can write it as $L_j = \sum_{j=1}^N a_{jk} \partial/\partial x_k$ with $a_{jk} \in \mathbb{C}$ and we denote write the symbol of L_j as

$$\widehat{L}_j(\xi) = i \sum_{j=1}^N a_{jk} \xi_k$$

for $\xi \in \mathbb{Z}^N$. Next, we state a condition that is sufficient for obtaining such bi-invariant representatives in every bi-degree.

Definition 5 We say that a CR subalgebra $\mathfrak{m} \subset \mathbb{C}t$ satisfies the **(DC)** condition if there exist a basis $\{L_1, \dots, L_n\}$ for \mathfrak{m} and constants $C, \rho > 0$ such that

$$\max_j |\widehat{L}_j(\xi)| \geq C(1 + |\xi|)^{-\rho}, \quad \forall \xi \in \mathbb{Z}^N.$$

The definition of the **(DC)** condition does not depend on the choice of basis, in fact, let $\{L'_1, \dots, L'_n\}$ be any other basis for \mathfrak{m} , then there are constants $a_{jk} \in \mathbb{C}$ such that $L'_j = \sum_{k=1}^n a_{jk} L_k$ and we have

$$|\widehat{L}'_j(\xi)| \leq \sum_{k=1}^n |a_{jk}| |\widehat{L}_k(\xi)| \leq nA \max_j |\widehat{L}_j(\xi)|$$

with $A = \max_{j,k} |a_{jk}|$. That is, there are constants $c, C > 0$ such that

$$c \max_j |\widehat{L}'_j(\xi)| \leq \max_j |\widehat{L}_j(\xi)| \leq C \max_j |\widehat{L}'_j(\xi)|$$

6.1 Fourier series on forms

Some of the computations in this section were inspired by [8] and [5].

For $u \in C^\infty(\mathbb{T}; \Lambda^{0,q})$, we write $u = \sum_{|I|=q} u_I \tau_I$ and since each u_I is a smooth function on \mathbb{T} , by using Fourier transform, we can write

$$u(x) = \sum_{\xi \in \mathbb{Z}^N} e^{i\xi x} \sum_{|I|=q} \widehat{u}_I(\xi) \tau_I = \sum_{\xi \in \mathbb{Z}^N} e^{i\xi x} \widehat{u}(\xi)$$

with

$$\widehat{u}(\xi) = \sum_{|I|=q} \widehat{u}_I(\xi) \tau_I.$$

We denote by $\|\cdot\|$ the value

$$\|\widehat{u}(\xi)\| = \max_{|J|=q} \{|\widehat{u}_J(\xi)|\}.$$

Since $u \in C^\infty(\mathbb{T}; \Lambda^{0,q})$, each u_I is smooth, so for each $\nu \in \mathbb{Z}_+$ there is $C_\nu > 0$ such that

$$\|\widehat{u}(\xi)\| \leq C_\nu(1 + |\xi|)^{-\nu}$$

for all $\xi \in \mathbb{Z}^N$.

Lemma 5 A $(0, q)$ -form $u \in C^\infty(\mathbb{T}; \Lambda^{0,q})$ is $\bar{\partial}_b$ -closed if, and only if,

$$\left(\sum_{k=1}^n \widehat{L}_k(\xi) \tau_k \right) \wedge \widehat{u}(\xi) = 0, \quad \forall \xi \in \mathbb{Z}^N. \tag{9}$$

Proof We have

$$\begin{aligned} \bar{\partial}_b u &= \sum_{|I|=q} \sum_{k=1}^n \sum_{\xi \in \mathbb{Z}^N} \widehat{L_k u_I}(\xi) e^{i\xi x} \tau_k \wedge \tau_I \\ &= \sum_{\xi \in \mathbb{Z}^N} \left(\sum_{k=1}^n \sum_{|I|=q} \widehat{L_k u_I}(\xi) \tau_k \wedge \tau_I \right) e^{i\xi x} \\ &= \sum_{\xi \in \mathbb{Z}^N} \left[\left(\sum_{k=1}^n \widehat{L_k}(\xi) \tau_k \right) \wedge \widehat{u}(\xi) \right] e^{i\xi x} \end{aligned}$$

From the last equality, we easily see that $\bar{\partial}_b u = 0$ if and only if $(\sum_{k=1}^n \widehat{L_k}(\xi) \tau_k) \wedge \widehat{u}(\xi) = 0$ for all $\xi \in \mathbb{Z}^N$. □

Notice that if f is a $\bar{\partial}_b$ -closed function on \mathbb{T} and m satisfied the **(DC)** condition, then f is constant. From now on, we assume that $q > 0$.

For $J = (j_1, \dots, j_q)$ and $\sigma = j_k$, we write $J \setminus \sigma$ to denote the multi-index $(j_1, \dots, \hat{j}_k, \dots, j_q)$ and let $\varepsilon_{\sigma, J}$ be the signature of the permutation $(\sigma, J \setminus \sigma)$.

Lemma 6 *Let $u \in C^\infty(\mathbb{T}; \Lambda^{0,q})$ be a $\bar{\partial}_b$ -closed and suppose that $\sum_{k=1}^n \widehat{L_k}(\xi) \tau_k \neq 0$ and let $\sigma \in \{1, \dots, n\}$ be such that $|\widehat{L_\sigma}(\xi)| = \max\{|\widehat{L_k}(\xi)| : k = 1, \dots, n\}$. Then, the $(q - 1)$ -form*

$$\widehat{v}(\xi) = \sum_{|J|=q; \sigma \in J} \varepsilon_{\sigma, J} \frac{1}{\widehat{L_\sigma}(\xi)} \widehat{u_J}(\xi) \tau_{J \setminus \sigma}$$

satisfies

$$\left(\sum_{k=1}^n \widehat{L_k}(\xi) \tau_k \right) \wedge \widehat{v}(\xi) = \widehat{u}(\xi) \tag{10}$$

and there exist $C > 0$ not depending on ξ such that

$$\|\widehat{v}(\xi)\| \leq \frac{C}{|\widehat{L_\sigma}(\xi)|} \|\widehat{u}(\xi)\|. \tag{11}$$

Proof Notice that Eq. (11) follows directly from the definition of $\widehat{v}(\xi)$. To prove Eq. (10), we adapt the proof of Lemma 2.1 of [5]. By definition, we have

$$\left(\sum_{k=1}^n \widehat{L_k}(\xi) \tau_k \right) \wedge \widehat{v}(\xi) = \left(\sum_{k=1}^n \widehat{L_k}(\xi) \tau_k \right) \wedge \sum_{|J|=q; \sigma \in J} \varepsilon_{\sigma, J} \frac{1}{\widehat{L_\sigma}(\xi)} \widehat{u_J}(\xi) \tau_{J \setminus \sigma}. \tag{12}$$

On the other hand, we can write

$$\tau_\sigma = \frac{1}{\widehat{L_\sigma}(\xi)} \sum_{k=1}^n \widehat{L_k}(\xi) \tau_k - \frac{1}{\widehat{L_\sigma}(\xi)} \sum_{k=1, k \neq \sigma}^n \widehat{L_k}(\xi) \tau_k$$

and now

$$\begin{aligned} \hat{u}(\xi) &= \sum_{|J|=q;\sigma \in J} \widehat{u}_J(\xi) \varepsilon_{\sigma,J} \tau_\sigma \wedge \tau_{J \setminus \sigma} + \sum_{|J|=q;\sigma \notin J} \widehat{u}_J(\xi) \tau_J \\ &= \sum_{|J|=q;\sigma \in J} \widehat{u}_J(\xi) \varepsilon_{\sigma,J} \left(\frac{1}{\widehat{L}_\sigma(\xi)} \sum_{k=1}^n \widehat{L}_k(\xi) \tau_k \right) \wedge \tau_{J \setminus \sigma} \\ &\quad + \sum_{|J|=q;\sigma \notin J} \widehat{u}_J(\xi) \tau_J - \sum_{|J|=q;\sigma \in J} \widehat{u}_J(\xi) \varepsilon_{\sigma,J} \left(\frac{1}{\widehat{L}_\sigma(\xi)} \sum_{k=1, k \neq \sigma}^n \widehat{L}_k(\xi) \tau_k \right) \wedge \tau_{J \setminus \sigma} \end{aligned} \tag{13}$$

We define

$$\hat{u}_\sigma(\xi) = \sum_{|J|=q;\sigma \in J} \widehat{u}_J(\xi) \varepsilon_{\sigma,J} \left(\frac{1}{\widehat{L}_\sigma(\xi)} \sum_{k=1}^n \widehat{L}_k(\xi) \tau_k \right) \wedge \tau_{J \setminus \sigma}$$

and

$$\hat{u}_{-\sigma}(\xi) = \sum_{|J|=q;\sigma \notin J} \widehat{u}_J(\xi) \tau_J - \sum_{|J|=q;\sigma \in J} \widehat{u}_J(\xi) \varepsilon_{\sigma,J} \left(\frac{1}{\widehat{L}_\sigma(\xi)} \sum_{k=1, k \neq \sigma}^n \widehat{L}_k(\xi) \tau_k \right) \wedge \tau_{J \setminus \sigma}.$$

Now we have $\hat{u}(\xi) = \hat{u}_\sigma(\xi) + \hat{u}_{-\sigma}(\xi)$. Since u is $\bar{\partial}_b$ -closed, it holds that $(\sum_{k=1}^n \widehat{L}_k(\xi) \tau_k) \wedge \hat{u}(\xi) = 0$ and, by construction, we have that

$$\left(\sum_{k=1}^n \widehat{L}_k(\xi) \tau_k \right) \wedge \hat{u}_\sigma(\xi) = 0$$

and thus

$$\left(\sum_{k=1}^n \widehat{L}_k(\xi) \tau_k \right) \wedge \hat{u}_{-\sigma}(\xi) = 0.$$

By expanding this last equality, and using the fact that $\tau_k \wedge \tau_J$ form a basis for all $(q + 1)$ -forms, we obtain that $\hat{u}_{-\sigma}(\xi) = 0$ and thus $\hat{u}(\xi) = \hat{u}_\sigma(\xi) = (\sum_{k=1}^n \widehat{L}_k(\xi) \tau_k) \wedge \widehat{v}(\xi)$. □

Lemma 7 *Let $u \in C^\infty(\mathbb{T}; \Lambda^{0,q})$ $\bar{\partial}_b$ -closed and suppose that \mathfrak{m} satisfies condition (DC). Then, the form*

$$u_* = \sum_{\xi \in \mathbb{Z}^N \setminus \{0\}} e^{i\xi x} \widehat{u}(\xi)$$

is $\bar{\partial}_b$ -exact.

Proof For each $\xi \in \mathbb{Z}^N$ with $\xi \neq 0$, by the condition (DC) we have $\sum_{k=1}^n \widehat{L}_k(\xi) \tau_k \neq 0$, and by Lemma 6, there exist $\widehat{v}(\xi)$ such that $(\sum_{k=1}^n \widehat{L}_k(\xi) \tau_k) \wedge \widehat{v}(\xi) = \widehat{u}(\xi)$ and so we can define $v = \sum_{\xi \in \mathbb{Z}^N \setminus \{0\}} e^{i\xi \cdot x}$. Notice that by (11) the form v is smooth and by construction we have $\bar{\partial}_b v = u_*$. □

If $[u]$ is a cohomology class in $H^{0,q}(T; \mathfrak{m})$, we have that if we define

$$u_0 = \widehat{u}(0) = \sum_{|J|=q} \widehat{u}_J(0) \tau_J,$$

then $[u] = [u_0]$. In fact, $u_* = u - u_0$ is $\bar{\partial}_b$ -exact by the last lemma and it follows that each cohomology class has a left-invariant representative. In particular, the cohomology groups $H^{0,q}(T; \mathfrak{m})$ are finite-dimensional.

7 Open problems

Here we list a few open problems that can give us some directions for future research.

1. The main tool we used in this project was the Leray–Hirsch theorem which can be understood as a restricted version of the Leray spectral sequence. What more could be obtained if the spectral sequence was used in full generality?
2. We imposed a strong condition to guarantee the existence of a cohomology extension. Is it possible to find weaker conditions that still guarantee the existence of cohomology extensions?
3. The classification theorem for CR algebras of maximal rank was very useful in the study of the related cohomology spaces. Therefore, it is only natural to ask: Is it possible to find a classification theorem for left-invariant CR structures in general, without assuming that they are of maximal rank?
4. Let G be a compact Lie group endowed with an elliptic Lie algebra \mathfrak{h} . We assume that there exist a maximal torus $T \subset G$ such that \mathfrak{h} can be decomposed as

$$\mathfrak{h} = \mathfrak{e} \oplus \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha \quad (14)$$

with \mathfrak{e} a bi-invariant elliptic structure over T . Since every elliptic structure is Levi-flat, the smooth bundle $T \rightarrow G \rightarrow G/T$ admits local trivializations that are compatible with the elliptic structures. We also have that every cohomology class on the torus T has a bi-invariant representative. Therefore, the proof of Theorem 1 can easily be adapted to this context. This raises the following question: what are the necessary and sufficient conditions so that left-invariant elliptic structures \mathfrak{h} admit a decomposition as in (14)?

Author Contributions The initial conceptualization of the project is due to MRJ. During the course of the research, both MRJ and HJ collaborated equally to obtain the presented results. The first draft was prepared MRJ. Then, both authors contributed in the writing process and thoroughly reviewed the manuscript.

Funding Open Access funding enabled and organized by Projekt DEAL. The second author was partially funded by FAPESP (Grants No. 19/09967-8 and 19/22981-0) and DFG (Grant No. JA 3453/1-1) during this work.

Declarations

Conflict of interest The authors declare no conflict of interest.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Alexandrov, B., Ivanov, S.: Vanishing theorems on Hermitian manifolds. *Differ. Geom. Appl.* **14**(3), 251–265 (2001)
2. Berhanu, S., Cordaro, P.D., Hounie, J.: *An Introduction to Involutive Structures*. New Mathematical Monographs, vol. 6. Cambridge University Press, Cambridge (2008)
3. Besse, A.L.: *Einstein Manifolds*. Classics in mathematics, Springer-Verlag, Berlin (2008). (**Reprint of the 1987 edition**)
4. Bor, G., Jacobowitz, H.: Left-invariant CR structures on 3-dimensional Lie groups. *Complex Anal. Synerg.* **7**(3), 23 (2021)
5. Bergamasco, A.P., Petronilho, G.: Global solvability of a class of involutive systems. *J. Math. Anal. Appl.* **233**(1), 314–327 (1999)
6. Charbonnel, J.-Y., Khalgui, H.O.: Classification des structures CR invariantes pour les groupes de Lie compacts. *J. Lie Theory* **14**(1), 165–198 (2004)
7. Cordaro, P.D.: Global hypoellipticity for $\bar{\partial}_b$ on certain compact three-dimensional CR manifolds. *Resenhas* **2**(4), 353–361 (1996)
8. Dattori da Silva, P.L., Meziani, A.: Cohomology relative to a system of closed forms on the torus. *Math. Nachr.* **289**(17–18), 2147–2158 (2016)
9. Gadea, P.M., Muñoz Masqué, J., Mykytyuk, I.V.: *Problem Books in Mathematics. Analysis and algebra on differentiable manifolds*, 2nd edn. Springer, London (2013). (**A workbook for students and teachers, With a foreword by Andrew Swann**)
10. Greenfield, S.J., Wallach, N.R.: Global hypoellipticity and Liouville numbers. *Proc. Am. Math. Soc.* **31**, 112–114 (1972)
11. Hall, B.: *Lie Groups, Lie Algebras, and Representations*, Volume 222 of Graduate Texts in Mathematics, 2nd edn. Springer, Cham (2015). (**An elementary introduction**)
12. Hilgert, J., Neeb, K.-H.: *Springer Monographs in Mathematics. Structure and geometry of Lie groups*, Springer, New York (2012)
13. Knapp, Anthony W.: *Lie Groups Beyond an Introduction*, volume 140 of Progress in Mathematics, 2nd edn. Birkhäuser Boston, Inc., Boston (2002)
14. Pittie, H.V.: The Dolbeault-cohomology ring of a compact, even-dimensional Lie group. *Proc. Indian Acad. Sci. Math. Sci.* **98**(2–3), 117–152 (1988)
15. Snow, Dennis M.: Invariant complex structures on reductive Lie groups. *J. Reine Angew. Math.* **371**, 191–215 (1986)
16. Spanier, E.H.: *Algebraic Topology*. Springer-Verlag, New York (1995). (**Corrected reprint of the 1966 original**)
17. Trèves, F.: *Hypo-analytic Structures*. Princeton mathematical series, vol. 40. Princeton University Press, Princeton (1992). (**Local theory**)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.