Proof that one can not Trisect an angle of 60 degrees with Straight Edge and Compass

Original Notes adopted from February 12, 2002 (Week 19)

F, $r \in F$, $r>0$ & if $\sqrt{r} \in F$,
$F(\sqrt{r}) = \{a + b\sqrt{r}: : a,b \in F\}$

Tower of number fields is a collection
$F_0 \subset F_1 \subset F_2 \subset .... \subset F_n$ of number fields such that
$F_0 = Q$ to $\forall$ i, there exists $r_i \in F_i$ such that $r > 0$, $\sqrt{r_i} \notin F_i$, & $F_{i+1} = F_i(\sqrt{r_i})$

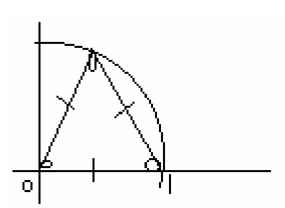
Collection of surds is U $\{F_n : F_n$ at top of a tower$\}$
S, S number field ("the surd field")

Set of Constructible Numbers:

We showed: C is a number field
Also, if $r \in C$, $r > 0$, then $\sqrt{r} \in C$.

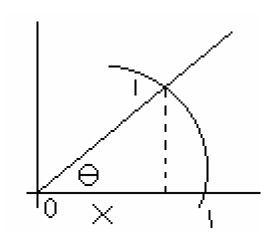
We showed: **Theorem: C = S.**

We'll show: Can't Trisect (with compass and straightedge) an angle of 60 degrees.
Can construct angle of 60 degrees.



If we could trisect that angle of 60 degrees, we could construct an angle of 20 degrees. We'll show that this is impossible.

Lemma 1: If angle θ(acute) is constructible, then cosθ is a constructible number.



$\cos\theta = x/1 = x$.
Constructible x from θ.
$\therefore \cos\theta$ constructible.

Suffices for:
Theorem: That 60 degrees not trisectable to show cos20 degrees not constructible.
I.e. Show cos20 degrees is not a surd.

Recall: $\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

$\cos3\theta = \cos(2\theta + \theta)$

$\cos3\theta = \cos2\theta\cos\theta - \sin2\theta\sin\theta$

$\cos3\theta = (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta$

$\cos3\theta = 2\cos^3\theta - \cos\theta - 2(1 - \cos^2\theta)\cos\theta$

$\cos3\theta = 4\cos^3\theta - 3\cos\theta$ $\theta = 20°$

$\cos3\theta = \cos60° = ½$ where $\theta = 20°$

$\therefore$ Get $½ = 4\cos^3\theta - 3\cos\theta$

$8(\cos20°)^3 - 6(\cos20°) - 1 = 0.$

If cos20° was in C, then x satisfies $x^3 - 3x - 1 = 0$.

If 60° trisectible, there would be a constructible root of $x^3 - 3x - 1 = 0$. We'll show:

1) Lemma 2: If a cubic equation with rational coefficients has a constructible root, then it has a rational root.

2) Lemma 3: $x^3 - 3x - 1 = 0$ has no rational root.

<u>To prove Lemma 2: We' ll need some preliminary Results</u>

a) Suffices to consider cubics with leading coefficient' s 1 (or else divide through it)

b)$(x - r_1)(x - r_2)(x - r_3)$ is a typical cubic with leading coefficients where r_1, r_2, r_3 are the (perhaps complex) roots.

$(x - r_1)(x - r_2)(x - r_3) = x^3 - r_1x^2 = r_2x^2 - r_3x^2 + ...$

$= x^3 - (r_1 + r_2\ r_3)x^2 + ...$

Note: Coefficients of x^2 is sum of the roots.

c) Thus the sum of the three roots of a cubic with rational coefficients is rational.

d) **Definition: If $a + b\sqrt{r} \in F(\sqrt{r})$, define the conjugate of $a + b\sqrt{r}$ to be $a - b\sqrt{r}$, and we use the notation.**

$$\overline{(a + b\sqrt{r})} = a - b\sqrt{r}.$$

Eg. $2/3 - 4\sqrt{3} \in Q(\sqrt{3})$

$$\overline{2/3 - 4\sqrt{3}} = 2/3 + 4\sqrt{3}.$$

e) The conjugate of a sum of two numbers is the sum of their conjugates.

Proof:
$$\begin{aligned}\overline{(a + b\sqrt{r}) + (c + d\sqrt{r})} &= \overline{(a+c) + (b+d)\sqrt{r}} \\ &= (a + c) - (b + d)\sqrt{r} \\ &= a - b\sqrt{r} + c - d\sqrt{r} \\ &= \\ &= \overline{a + b\sqrt{r}} + \overline{c + d\sqrt{r}}\end{aligned}$$

f) The conjugate of a product of two numbers is the product of the conjugate

$\overline{(a + b\sqrt{r})\ (c + d\sqrt{r})}\quad \overline{(ac + bdr) + (bd + cb)\sqrt{r}}$

$= (ac + bdr) - (ad + cb)\sqrt{r}$

Also, $(a - b\sqrt{r})(c - d\sqrt{r}) = ac + bdr - (ad + cb)\sqrt{r}$ = above

g) Corollary: For any natural number k, $\overline{(a + b\sqrt{r})}^{k} = (a - b\sqrt{r})^{k}$

h) Theorem: If p, polynomial with rational coefficients & if $p(a + b\sqrt{r}) = 0$ for $a + b\sqrt{r}$ in a tower,
then $p(a - b\sqrt{r}) = 0$.
(ie. the conjugate of a root is also a root).

Proof:
If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots a_1 x + a_0$, $a_i \in Q$.
$0 = p(a + b\sqrt{r})$

$\overline{0} = 0 = \overline{p(a + b\sqrt{r})} = \overline{a_n(a + b\sqrt{r})^n + a_{n-1}(a + b\sqrt{r})^{n-1} + \ldots + a_1(a + b\sqrt{r}) + a_0}$

$= \overline{a_n(a + b\sqrt{r})^n} + \overline{a_{n-1}(a + b\sqrt{r})^{n-1}} + \overline{a_1(a + b\sqrt{r})} + \overline{a_0}$
$= a_n(a - b\sqrt{r})^n + a_{n-1}(a - b\sqrt{r})^{n-1} + \ldots + a_1(a - b\sqrt{r}) + a_0$
$= p(a - b\sqrt{r})$, so $p(a - b\sqrt{r}) = 0$.

i) Theorem: If a cubic equation with rational coefficients has a constructible root, then it has a rational root.

Proof: Recall that the sum of these roots is rational.
C = S, So given there exists a root x_0 in some F_k where $Q = F_0 \subset F_1 \subset F_2 \subset \ldots \subset F_k$
is a tower with $F_i = F_{i-1}(\sqrt{r_i})$
$x_0 = a + b_0\sqrt{r_{k-1}}$ $r_{k-1} \in F_{k-1}$, $a_0, b_0 \in F_{k-1}$

Assume we choose the shortest tower containing x_0 ie) $b_0 \neq 0$.
(If x_0 were in Q, we're done).

By above, $a_0 - b_0\sqrt{r_{k-1}}$ is a root. Let s be the 3rd root such that
$(a_0 + b_0\sqrt{r_{k-1}}) + (a_0 - b_0\sqrt{r_{k-1}}) = q$, some $q \in Q$,
$s + 2a_0 = q$
$s = q - 2a_0 \in F_{k-1}$

Thus if root in some $F_k \neq Q$, there is root in F_{k-1}
if $F_{k-1} \neq Q$, apply again, get root in F_{k-2} .... Etc$\rightarrow$ until there exists a root in Q (smallest Tower is Q)

Thus 60° not trisectible if we can show:

Lemma: $x^3 - 3x - 1 = 0$ doesn' t have a rational root.

Proof: Suppose x = m/n, is a natural root. m,n, integers, in lowest terms.

$\left(\frac{m}{n}\right)^3 - 3\left(\frac{m}{n}\right) - 1 = 0$

$$\frac{m^3}{n^3} - 3\frac{m}{n} - 1 = 0$$

$m^3 - 3mn^2 - n^3 = 0.$

If $p|n$, $p|(3mn^2 + n^3) \Rightarrow p|m^3 \Rightarrow p|m$

(p prime, since m/n in lowest terms, no such p $\therefore$ +/- 1)
If q prime, and $q|m$, then $q|m$, then $q|(m^3 - 3mn^2)$, so $q|n^2$
So $q|n$. m,n relatively prime $\Rightarrow m = +/- 1$.

$m,n = +/- 1 \Rightarrow m/n = +/- 1$.
$\therefore$ If 60° trisectible, $x^3 - 3x - 1 = 0$ for x = 1, or x = -1.

$1^3 - 3(1) - 1 = -3$
$(-1)^3 - 3(-1) - 1 = 1$ $\therefore$ Not trisectible

<u>Definition</u>: A regular polygon is a polygon all of whose sides are equal and all of whose angles are equal.
n = n^0 of sides
n = 3: equilateral triangle ---> Constructible
n = 4: Square ----> Constructible
n = 5 Regular Pentagon ----> ?